

Properties of the limit shape for some last passage growth models in random environments

By

Hao Lin

A DISSERTATION SUBMITTED IN PARTIAL FULFILLMENT OF THE
REQUIREMENTS FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY
(MATHEMATICS)

at the

UNIVERSITY OF WISCONSIN – MADISON

2011

Abstract

We study directed last-passage percolation on the planar square lattice whose weights have general distributions, or equivalently, queues in series with general service distributions. Each row of the last-passage model has its own randomly chosen weight distribution. We first show the existence of the limiting time constant and list its properties. Next we study the problem for models with Bernoulli and exponential weights, for which we already have more precise results. We then present some universality results about the limiting time constant close to the boundary of the quadrant. Close to the y -axis, where the number of random distributions averaged over stays large, the limiting time constant takes the same universal form as in the homogeneous model. But close to the x -axis we see the effect of the tail of the distribution of the random environment. In particular we will give some estimates of the upper bound in this case.

Acknowledgements

I would like to express my deepest gratitude to my advisor, Prof. Timo Seppäläinen, for his guidance and patience throughout the years. He not only has taught me how to make progress in mathematical research, but also demonstrated me a good example of combining both rigorous scholarship and accessibility to audience in his teaching, academic talks and papers. This dissertation would not have been possible without his continued feedback and encouragement.

I also appreciate all valuable suggestions from the committee members, Prof. Benedek Valko, Prof. David Anderson, Prof. Jordan Ellenberg and Prof. Gregorio Moreno-Flores. Great thanks for your time and comments on my work.

I would like to thank Prof. Tom Kurtz and Prof. David Griffeath, from whom I took several probability courses. You are the first teachers who showed me a beautiful picture of probability theory and motivated me to major in it eventually.

I would like to thank my colleagues Matthew Joseph, Rohini Kumar and Nicos Georgiou for sharing their experience and wisdom with me. I learned a lot from you all. Also thanks for my dear friends in the department: Jingwei Guo, Anakewit Boonkasame, Hwan Lee, Gabriel Pretel and many many other names. You guys have made my life colorful here!

I am grateful to the department staff: Sharon Paulson, Mary Rice, Vicky Whelan, Joan Wendt Yvonne Nagel and Mike Grenie. Thanks for tolerating my endless questions and requests for assistance.

At last, indescribable thanks to my parents. Your love is the meaning of my life!

Contents

Abstract	i
Acknowledgements	ii
1 Introduction	1
2 The existence and properties of the limiting time constant	10
2.1 The last-passage percolation model	10
2.2 The existence and properties	11
3 Results for Bernoulli and exponential models	25
3.1 Bernoulli models with strict-weak paths in a random environment	25
3.2 Limiting shapes for exponential models	37
3.2.1 Estimate of $\Psi_G(\alpha, 1)$	37
3.2.2 Estimate of $\Psi_G(1, \alpha)$	45
4 Universality results	51
4.1 Limiting shape near the y -axis	51
4.1.1 Proof of Theorem 4.1.1	52
4.1.2 An improvement on the error $o(\sqrt{\alpha})$	69
4.2 Estimates for limiting shape near x -axis	71
Bibliography	89

Chapter 1

Introduction

This paper studies the limit shapes of some last-passage percolation models in random environments. Specifically, we will first derive the hydrodynamic limit of the last-passage time for the corner growth model with exponential weights and for two Bernoulli models with different rules for admissible paths. Next, we will present some universality results for the limit shape for a broader range of underlying distributions.

We begin by introducing the corner growth model through its queueing interpretation. Consider service stations in series, labeled $0, 1, 2, \dots, \ell$, each with unbounded waiting room and first-in first-out (FIFO) service discipline. Initially customers $0, 1, 2, \dots, k$ are queued up at server 0. At time $t = 0$ customer 0 begins service with server 0. Each customer moves through the system of servers in order, joining the queue at server $j + 1$ as soon as service with server j is complete. After customer i departs server j , server j starts serving customer $i + 1$ immediately if $i + 1$ has been waiting in the queue, or then waits for customer $i + 1$ to arrive from station $j - 1$. Customers stay ordered throughout the process. Let $X(i, j)$ be the service time that customer i needs at station j , and $T(k, \ell)$ the time when customer k completes service with server ℓ .

Asymptotics for $T(k, \ell)$ as k and ℓ get large have been investigated a great deal in the past two decades. A seminal paper by Glynn-Whitt [6] studied the case of i.i.d. $\{X(i, j)\}$. They took advantage of the connection with directed last-passage percolation

given by the identity

$$T(k, \ell) = \max_{\pi} \sum_{(i,j) \in \pi} X(i, j). \quad (1.0.1)$$

In this model, $X(i, j)$ is a random weight assigned to the point (i, j) . The maximum is taken over non-decreasing nearest-neighbor lattice paths $\pi \subseteq \mathbb{Z}_+^2$ from $(0, 0)$ to (k, ℓ) that are of the form $\pi = \{(0, 0) = (x_0, y_0), (x_1, y_1), \dots, (x_{k+\ell}, y_{k+\ell}) = (k, \ell)\}$ where $(x_i, y_i) - (x_{i-1}, y_{i-1}) = (1, 0)$ or $(0, 1)$. Below is a picture of an admissible path from $(0, 0)$ to $(4, 3)$:

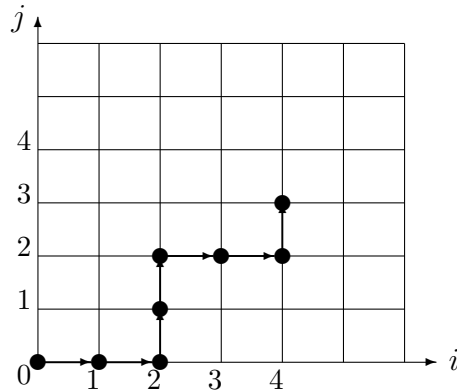


Figure 1: An admissible path to $(4, 3)$.

It is easy to see that both the queueing setting and (1.0.1) satisfy the following recursive relationship for positive k and ℓ :

$$T(k, \ell) = \max\{T(k-1, \ell), T(k, \ell-1)\} + X(k, \ell). \quad (1.0.2)$$

Therefore if the process $\{X(i, j)\}$ in the last-passage model has the same distribution as $\{X(i, j)\}$ in the queueing model, then $\{T(k, \ell)\}$ defined in (1.0.1) and in the queueing model have the same distribution, too. (1.0.1) together with earlier references to this

observation can be found in [6] (see Prop. 2.1). This particular last-passage model is also known as the *corner growth model*.

Next we add a random environment to both the queueing and last-passage percolation models. The environment is a sequence $\{F_j\}_{j \in \mathbb{Z}_+}$ of probability distributions, generated by a probability measure-valued ergodic or i.i.d. process with distribution \mathbb{P} . Given the sequence $\{F_j\}_{j \in \mathbb{Z}_+}$, we assume that the variables $\{X(i, j)\}$ are independent and $X(i, j)$ has distribution F_j . In the queueing picture this means that for each $j \in \mathbb{Z}_+$ the service times $\{X(i, j) : i \in \mathbb{Z}_+\}$ at service station j have common distribution F_j , and at the outset the distributions $\{F_j\}_{j \in \mathbb{Z}_+}$ themselves are chosen randomly according to some given law \mathbb{P} . Obviously the labels “customer” and “server” are interchangeable because we can switch around the roles of the indices i and j . In the last-passage percolation model, the random environment means that weights assigned to points on the j -th row follow F_j .

Although (1.0.2) is simple and clear, it does not suffice to provide much information about $T(k, \ell)$ when k and ℓ are large. In fact, it is not very realistic to ask what is the distribution of $T(k, \ell)$. Instead, we let k and ℓ go to infinity and scale $T(k, \ell)$ in a proper way. The asymptotic regime we consider for $T(k, \ell)$ is the *hydrodynamic* one where k and ℓ are both of order n and n is taken to ∞ . Under some moment assumptions standard subadditive considerations and approximations imply the existence of the deterministic limit for all positive real numbers x and y :

$$\Psi(x, y) = \lim_{n \rightarrow \infty} n^{-1} T(\lfloor nx \rfloor, \lfloor ny \rfloor).$$

We will also verify some properties of $\Psi(x, y)$ in Section 2.2: homogeneity, concavity

and continuity.

Only in the case where the distributions F_j are exponential or geometric has it been possible to describe explicitly the limit Ψ . This is the case of $\cdot/M/1$ queues in series, which in terms of interacting particle systems is the same as studying either the totally asymmetric simple exclusion process or the zero-range process with constant jump rate. For i.i.d. exponential $\{X(i, j)\}$ with rate 1, the limit $\Psi(x, y) = (\sqrt{x} + \sqrt{y})^2$ was first derived by Rost [17] in a seminal paper on hydrodynamic limits of asymmetric exclusion processes.

The random environment model with exponential F_j 's was studied in [1, 12, 20]. The exact $\Psi(x, y)$ can be described implicitly. It depends on the specific distribution of the exponential rates. In Section 3.2 we will see some explicit estimates of $\Psi(\alpha, 1)$ and $\Psi(1, \alpha)$ when α is small. These two quantities have different behaviors and will be discussed in further details.

Let us now set aside the queueing motivation and consider the last-passage model on the first quadrant \mathbb{Z}_+^2 of the planar integer lattice, defined by the nondecreasing lattice paths and the random weights $\{X(i, j)\}$. For the queueing application it is natural to assume the weights nonnegative, but in the general last-passage situation there is no reason to restrict to nonnegative weights.

The ideal limit shape result would have some degree of universality, that is, apply to a broad class of distributions. Such results have been obtained only close to the boundary: in [13] Martin showed that in the i.i.d. case, under suitable moment hypotheses and as $\alpha \searrow 0$,

$$\Psi(1, \alpha) = \mu + 2\sigma\sqrt{\alpha} + o(\sqrt{\alpha}), \quad (1.0.3)$$

where μ and σ^2 are the common mean and variance of the weights $X(i, j)$. The $o(\sqrt{\alpha})$ term in the statement means that $\lim_{\alpha \searrow 0} \alpha^{-1/2} [\Psi(\alpha, 1) - \mu - 2\sigma\sqrt{\alpha}] = 0$. In the i.i.d. case Ψ is symmetric so the same holds for $\Psi(\alpha, 1)$.

Our goal is to find the form Martin's result takes in the random environment setting. Ψ is no longer necessarily symmetric since the distribution of the array $\{X(i, j)\}$ is not invariant under transposition. So we must ask the question separately for $\Psi(1, \alpha)$ and $\Psi(\alpha, 1)$.

It turns out that for $\Psi(\alpha, 1)$, where the number of rows stays large relative to the number of columns, the fluctuations of the environment average out to the degree that our result in Theorem 4.1.1 is essentially identical to Martin's result in the homogeneous environment. We still have $\Psi(\alpha, 1) = \mu + 2\sigma\sqrt{\alpha} + o(\sqrt{\alpha})$ as $\alpha \searrow 0$, where now μ is the average of the “quenched” mean and σ^2 is the average of the “quenched” variance. That is, if we let $\mu_0 = \int x dF_0(x)$ and $\sigma_0^2 = \int (x - \mu_0)^2 dF_0(x)$ denote the mean and variance of the random distribution F_0 , and \mathbb{E} expectation under \mathbb{P} , then $\mu = \mathbb{E}(\mu_0)$ and $\sigma^2 = \mathbb{E}(\sigma_0^2)$.

There is some evidence that we can do better than $o(\sqrt{\alpha})$ for the error term. If $\{F_j\}$ is a sequence of exponential distributions, the result

$$\Psi(\alpha, 1) = \mu + 2\sigma\sqrt{\alpha} + O(\alpha)$$

can be shown. For general distributions with uniform boundedness, one can achieve $o(\alpha^{\frac{3}{5}-\varepsilon})$ for any $\varepsilon > 0$. Although not yet proved, we conjecture that $O(\alpha)$ should be the answer even for general $\{F_j\}$. The first step should be to prove this for the homogeneous case.

The case $\Psi(1, \alpha)$ does not possess a clean result such as the one above. Even though we are studying the deterministic limit obtained *after* n has been taken to infinity, we see an effect from the tail of the distribution of the quenched mean μ_0 . We illustrate this with the case of exponential $\{F_j\}$. Now the number $n\alpha$ of distributions F_j is small compared to the number n of weights $X(i, j)$ in each row, hence the fluctuations among the F_j 's become prominent. The effect comes in two forms: first, the leading term is no longer the averaged mean μ but the maximal mean μ^* . Second, if large values among the row means are rare, the order of the α -dependent correction is smaller than the $\sqrt{\alpha}$ seen above and this order of magnitude depends on the tail of the distribution of μ_0 . As an exponent characterizing this tail changes, we can see a phase transition of sorts in the power of α , with a logarithmic correction at the transition point.

Intuitively, the above two phenomena suggest that when very few rows compared to columns are available, the optimal path makes most of its horizontal movement along the rows with large means very close to μ^* . Therefore when large means are rare, there are not many candidates for the optimal path, so $\Psi(1, \alpha) - \mu^*$ tends to be smaller. The other extreme is that all means are μ^* , i.e. they are equal. In this case we can recall what happens in the homogeneous case and guess the first α -dependent term may be $\sqrt{\alpha}$. We will verify this idea in the exponential model, and derive an upper bound on $\Psi(1, \alpha)$ that gives the correction of order $\sqrt{\alpha}$ as well for general distributions under sufficient conditions.

The key idea in proving universality results in this paper is to compare limiting time constant in models with general distributions to that in models with normal distributions. For this purpose we need to quantify the difference between $\Psi_F(x, y)$ and $\Psi_G(x, y)$ for two processes $\{F_j\}_{j \in \mathbb{Z}_+}$ and $\{G_j\}_{j \in \mathbb{Z}_+}$. An example of this is Lemma 4. In the proof

we use as auxiliary results bounds on the limits of last-passage models with Bernoulli weights .

It is worth noting that with Bernoulli weights the limiting time constant $\Psi(x, y)$ has not been derived for the standard corner growth model. $\Psi(x, y)$ can be solved in a model with Bernoulli weights when the path geometry is altered suitably. The model we take up is the one where the paths are weakly increasing in one coordinate but strictly in the other. There are two cases, depending on which coordinate is required to increase strictly. If we require the x -coordinate to increase strictly then an admissible path $\{(x_0, y_0), (x_1, y_1), \dots, (x_m, y_m)\}$ satisfies

$$x_{i+1} - x_i = 1 \text{ and } y_0 \leq y_1 \leq \dots \leq y_m. \quad (1.0.4)$$

We give a figure below showing a possible path:

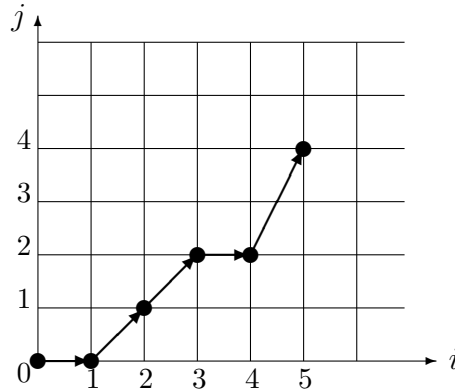


Figure 2: An admissible path to $(5, 4)$ with x -coordinate strictly increasing.

The other case interchanges x and y . These cases have to be addressed separately because the random environment attached to rows makes the model asymmetric, i.e.

the value of $\Psi(x, y)$ changes when we interchange the two coordinates. The sum of these two last-passage values gives a bound for the case where neither coordinate is required to increase strictly in each step.

We derive the exact limit constants for Bernoulli models with both types of “strict/weak” paths. For one of them this has been done before by Gravner, Tracy and Widom [9]. Their proof utilizes the fact that the distribution of $T(k, \ell)$ is a symmetric function of the environment in the sense that it is not affected if we interchange the distributions in any two rows (at least for the particular Bernoulli case they study). The proof here is completely different. It is based on the idea in [19] where the limit for the homogeneous case was derived: the last-passage model is coupled with a particle system whose invariant distributions can be written down explicitly, and then through some convex analysis the speed of a tagged particle yields the explicit limit of the last-passage model. This same approach can be adapted to the random environment case so that results in [19] can be generalized.

Further remarks on the literature. In [6], a different asymptotic regime given by $\frac{1}{n^{\frac{1+a}{2}}}T(\lfloor n^a x \rfloor, n)$ with $0 < a < 1$ was studied in the homogeneous model. They derived an asymptotic result for the above quantity when the underlying distribution has an exponentially decaying tail. It would be interesting to see whether the result can be generalized to the random environment case.

Many papers also addressed questions of fluctuations. For the last-passage model with i.i.d. exponential or geometric weights, the distributional limit with fluctuations of order $n^{1/3}$ and limit given by the Tracy-Widom GUE distribution was proved by Johansson [10]. As for the shape, universality has been achieved only close to the boundary, by Baik-Suidan [2] and Bodineau-Martin [3].

Fluctuations of the Bernoulli model with strict/weak paths and homogeneous weights were derived first in [11] and later also in [7]. For the model in a random environment fluctuation limits appear in [9, 8].

On the lattice \mathbb{Z}_+^2 we can imagine three types of nondecreasing paths: (i) weak-weak: both coordinates required to increase weakly, the type used in (1.0.1); (ii) strict-weak: one coordinate increases strictly, as above in (1.0.4); and (iii) strict-strict: both coordinates increase strictly so an admissible path $\{(x_0, y_0), (x_1, y_1), \dots, (x_m, y_m)\}$ satisfies $x_0 < \dots < x_m$ and $y_0 < \dots < y_m$. As mentioned, with Bernoulli weights the strict-weak case is solvable but the weak-weak case appears harder. The third case, strict-strict, is also solvable with Bernoulli weights. The shape was derived in [18] and recent work on this model appears in [5].

Organization of the paper. We begin by introducing the last-passage time percolation model in Chapter 2 and verify the existence of the limiting time constant in Section 2.2. Next we present some results specifically for Bernoulli models (Section 3.1) and exponential models (Section 3.2). Then we will show universality theorems on the shape close to the boundary in Chapter 4: in Section 4.1 we present Theorem 4.1.1 on $\Psi(\alpha, 1)$ and in Section 4.2 we have some estimates on $\Psi(1, \alpha)$.

Some frequently used notation. We write

$$\operatorname{ess\,sup}_{\mathbb{P}} f = \inf\{s \in \mathbb{R} : \mathbb{P}(f > s) = 0\}$$

for the essential supremum of a function f under a measure \mathbb{P} . $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$, $\mathbb{N} = \{1, 2, 3, \dots\}$. $I(A)$ is the indicator function of event A .

Chapter 2

The existence and properties of the limiting time constant

2.1 The last-passage percolation model

We give a precise definition of the last-passage model in a random environment. Let \mathbb{P} be a stationary, ergodic probability measure on the space $\mathcal{M}_1(\mathbb{R})^{\mathbb{Z}_+}$ of sequences of Borel probability distributions on \mathbb{R} . \mathbb{E} denotes expectation under \mathbb{P} . For some of the results in the following chapters \mathbb{P} will be further assumed to be an i.i.d. product measure. A realization of the distribution-valued process under \mathbb{P} is denoted by $\{F_j\}_{j \in \mathbb{Z}_+}$. This is the environment. Given $\{F_j\}$, the weights $\{X(z) : z \in \mathbb{Z}_+^2\}$ are independent real-valued random variables with marginal distributions $X(i, j) \sim F_j$ for $(i, j) \in \mathbb{Z}_+^2$. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be the probability space on which all variables $\{F_j, X(i, j)\}$ are defined, and denote expectation under \mathbf{P} by \mathbf{E} .

A (weakly) nondecreasing path is a sequence of points $z_0 = (x_0, y_0), z_1 = (x_1, y_1), \dots, z_m = (x_m, y_m)$ in \mathbb{Z}_+^2 that satisfy $x_0 \leq x_1 \leq \dots \leq x_m$, $y_0 \leq y_1 \leq \dots \leq y_m$, and $|x_{i+1} - x_i| + |y_{i+1} - y_i| = 1$. For $z_1, z_2 \in \mathbb{Z}_+^2$ with $z_1 \leq z_2$ (coordinatewise ordering), let $\Pi[z_1, z_2]$ be the set of nondecreasing paths from z_1 to z_2 . Whether the endpoints z_1 and z_2 are included in the path makes no difference to the limit results below, but to

be precise let us include z_1 and exclude z_2 , so that we can run a subadditive argument later.

Remark 1. *We included the endpoint in the definition above Figure 1 because we wanted it to be consistent with (1.0.2).*

Hereafter we will always exclude the endpoint when talking about a path between two points.

The last-passage time $T(z_1, z_2)$ from z_1 to z_2 is defined by

$$T(z_1, z_2) = \max_{\pi \in \Pi[z_1, z_2)} \sum_{z \in \pi} X(z).$$

When $z_1 = 0$ abbreviate $\Pi(z) = \Pi[0, z)$ and $T(z) = T(0, z)$.

$T(z)$ is a random variable that depends on the underlying distributions, and will be quite complicated as z moves far away from the origin. However, the following quantity, known as the limiting time constant, exists under proper conditions and provide information about the last-passage time

$$\Psi(x, y) = \lim_{n \rightarrow \infty} \frac{1}{n} T(\lfloor nx \rfloor, \lfloor ny \rfloor).$$

2.2 The existence and properties

We now give a set of sufficient conditions for the aforementioned limit to exist. Put these three assumptions on the model:

$$\mathbf{E}|X(z)| < \infty, \tag{2.2.1}$$

$$\int_0^\infty \left\{1 - \mathbb{E}(F_0(x))\right\}^{1/2} dx < \infty, \quad (2.2.2)$$

and

$$\int_0^\infty \operatorname{ess\,sup}_{\mathbb{P}}(1 - F_0(x)) dx < \infty. \quad (2.2.3)$$

We start with these assumptions and consider the existence of $\Psi(x, y)$.

Proposition 2.1. *Assume \mathbb{P} is ergodic and satisfies (2.2.1), (2.2.2) and (2.2.3). Then for all $(x, y) \in (0, \infty)^2$ the last-passage time constant*

$$\Psi(x, y) = \lim_{n \rightarrow \infty} \frac{1}{n} T(\lfloor nx \rfloor, \lfloor ny \rfloor) \quad (2.2.4)$$

exists as a limit both \mathbf{P} -almost surely and in $L^1(\mathbf{P})$. Furthermore, $\Psi(x, y)$ is a homogeneous, concave and continuous function on $(0, \infty)^2$.

Assumption (2.2.2) is also used for the constant distribution case, see (2.5) in [13]. Some further control along the lines of assumption (2.2.3) is required for our case. For example, suppose $1 - F_j(x) = e^{-\xi_j x}$ for random $\xi_j \in (0, \infty)$. Then (2.2.3) holds iff $\operatorname{ess\,inf}_{\mathbb{P}}(\xi_0) > 0$. If the distribution of ξ_0 is not bounded away from zero, $n^{-1}T(n, n) \rightarrow \infty$ because we can simply collect all the weights from the row with minimal ξ_j among $\{\xi_0, \dots, \xi_n\}$. However, assumption (2.2.2) can be satisfied without bounding ξ_0 away from zero.

Proof. We first prove the theorem for integer pairs (x, y) , and then extend to rational numbers and finally real numbers.

Step 1: consider $(x, y) \in \mathbb{N}^2$. Set $Z_{m,n} = -T((mx, my), (nx, ny))$ for $0 \leq m < n$, and verify that under the distribution \mathbf{P} , $Z_{m,n}$ satisfies assumptions (i), (ii) and

(iii) in Liggett's version of the subadditive ergodic theorem [4, p. 358]. In particular, $Z_{0,m} + Z_{m,n} \geq Z_{0,n}$, $\{Z_{nk,(n+1)k}, n \geq 1\}$ is ergodic for each k , and the distribution of $\{Z_{m,m+k}, k \geq 1\}$ does not depend on m .

We need to work harder on condition (iv), i.e. we need to show $\mathbf{E}Z_{0,1}^+ < \infty$ and for each n , $\mathbf{E}Z_{0,n} \geq \gamma n$ for some $\gamma > -\infty$. It is easy to see

$$\mathbf{E}Z_{0,1}^+ \leq \mathbf{E}|T((0,0), (x,y))| \leq \mathbf{E} \sum_{0 \leq i \leq x, 0 \leq j \leq y} |X(i,j)| = (x+1)(y+1)\mathbf{E}|X(0,0)| < \infty.$$

Next we show $\mathbf{E}Z_{0,n} \geq \gamma n$ for some $\gamma > -\infty$ under (2.2.2) and (2.2.3). This is trivially true for a Bernoulli model where given $\{F_j\}$ the weights have marginal distributions

$$P(X(i,j) = 1) = 1 - F_j(u) = 1 - P(X(i,j) = 0). \quad (2.2.5)$$

Therefore this Bernoulli model satisfies all conditions of the Subadditive ergodic theorem, and $\Psi_{Ber[1-F(u)]}(x,y)$, the limiting time constant, is well-defined $\mathbf{P} - a.s.$ and in $L^1(\mathbf{P})$ for $(x,y) \in \mathbb{N}^2$. We will see an upper bound (3.1.8) for $\Psi(x,y)$ of the Bernoulli model in the next chapter. We use it here without proof in the following calculation:

$$\begin{aligned} \frac{1}{n}\mathbf{E}Z_{0,n} &\geq -\frac{1}{n}\mathbf{E} \max_{\pi \in \Pi(nx,ny)} \sum_{z \in \pi} X(z)_+ = -\frac{1}{n}\mathbf{E} \max_{\pi \in \Pi(nx,ny)} \sum_{z \in \pi} \int_0^\infty I(X(z) > u) du \\ &\geq -\frac{1}{n}\mathbf{E} \int_0^\infty \max_{\pi \in \Pi(nx,ny)} \sum_{z \in \pi} I(X(z) > u) du \\ &= -\int_0^\infty \sup_n \frac{1}{n}\mathbf{E} \max_{\pi \in \Pi(nx,ny)} \sum_{z \in \pi} I(X(z) > u) du = -\int_0^\infty \Psi_{Ber[1-F(u)]}(x,y) du \\ &\geq -(y + 4\sqrt{xy}) \int_0^\infty \sqrt{1 - \mathbb{E}F_0(u)} du - x \int_0^\infty (1 - \operatorname{ess\,inf}_{\mathbb{P}} F_0(u)) du. \end{aligned} \quad (2.2.6)$$

Here $I(A)$ is the indicator function of event A . By assumptions (2.2.2) and (2.2.3), $\mathbf{E}Z_{0,n} \geq n\gamma$ for a constant $\gamma > -\infty$. These estimates justify the application of the subadditive ergodic theorem. So now for $(x, y) \in \mathbb{N}^2$, we can define the following \mathbf{P} -a.s. and $L^1(\mathbf{P})$ limit

$$\Psi(x, y) = \lim_{n \rightarrow \infty} \frac{1}{n} T(nx, ny).$$

Step 2: take $(x, y) \in (\mathbb{Q} \cap (0, \infty))^2$. Let $x = \frac{x_1}{x_2}$ and $y = \frac{y_1}{y_2}$ be in their reduced forms, i.e. x_i and y_i are positive for $i = 1, 2$ and $\gcd(x_1, x_2) = \gcd(y_1, y_2) = 1$. Let k be the least common multiple of x_2 and y_2 , so $(kx, ky) \in \mathbb{N}^2$.

For every positive integer n , write $n = Mk + r$ for integers M and r such that $0 \leq r \leq k - 1$. Then we have

$$Mkx \leq \lfloor nx \rfloor \leq (M+1)kx, \quad Mky \leq \lfloor ny \rfloor \leq (M+1)ky.$$

So if we denote $z_1(n) = (Mkx, Mky)$ and $z_2(n) = ((M+1)kx, (M+1)ky)$, we have the following inequalities from superadditivity:

$$T(z_1(n)) + T((z_1(n), (\lfloor nx \rfloor, \lfloor ny \rfloor))) \leq T(\lfloor nx \rfloor, \lfloor ny \rfloor) \leq T(z_2(n)) - T((\lfloor nx \rfloor, \lfloor ny \rfloor), z_2(n))$$

Obviously,

$$T(z_1(n), (\lfloor nx \rfloor, \lfloor ny \rfloor)) \geq - \max_{\pi \in \Pi[z_1(n), z_2(n)]} \sum_{z \in \pi} |X(z)|,$$

and this leads to

$$T(z_1(n)) - \max_{\pi \in \Pi[z_1(n), z_2(n)]} \sum_{z \in \pi} |X(z)| \leq T(\lfloor nx \rfloor, \lfloor ny \rfloor). \quad (2.2.7)$$

Let $\varepsilon > 0$ be any small positive number,

$$\begin{aligned}
& \sum_{n=1}^{\infty} \mathbf{P} \left(\max_{\pi \in \Pi[z_1(n), z_2(n)]} \sum_{z \in \pi} |X(z)| \geq n\varepsilon \right) \\
&= \sum_{n=1}^{\infty} \mathbf{P} \left(\max_{\pi \in \Pi[0, (kx, ky)]} \sum_{z \in \pi} |X(z)| \geq n\varepsilon \right) \\
&\leq \frac{1}{\varepsilon} \int_0^{\infty} \mathbf{P} \left(\max_{\pi \in \Pi[0, (kx, ky)]} \sum_{z \in \pi} |X(z)| \geq x \right) dx \tag{2.2.8} \\
&= \frac{1}{\varepsilon} \mathbf{E} \left(\max_{\pi \in \Pi[0, (kx, ky)]} \sum_{z \in \pi} |X(z)| \right) \\
&\leq \frac{1}{\varepsilon} (kx + 1)(ky + 1) \mathbf{E}|X(0, 0)| < \infty.
\end{aligned}$$

By Borel-Cantelli Lemma, $\frac{1}{n} \max_{\pi \in \Pi[z_1(n), z_2(n)]} \sum_{z \in \pi} |X(z)| \rightarrow 0$ $\mathbf{P} - a.s.$ Dividing through by n and taking limit in (2.2.7) gives that

$$\frac{1}{k} \Psi(kx, ky) = \lim_{n \rightarrow \infty} \frac{1}{n} T(z_1(n)) \leq \liminf_{n \rightarrow \infty} \frac{1}{n} T(\lfloor nx \rfloor, \lfloor ny \rfloor) \quad \mathbf{P} - a.s.$$

Similarly, we can show the other direction

$$\limsup_{n \rightarrow \infty} \frac{1}{n} T(\lfloor nx \rfloor, \lfloor ny \rfloor) \leq \lim_{n \rightarrow \infty} \frac{1}{n} T(z_2(n)) = \frac{1}{k} \Psi(kx, ky) \quad \mathbf{P} - a.s.$$

This shows that

$$\lim_{n \rightarrow \infty} \frac{1}{n} T(\lfloor nx \rfloor, \lfloor ny \rfloor) = \frac{1}{k} \Psi(kx, ky) \quad \mathbf{P} - a.s.$$

The definition of $\Psi(x, y)$ has now been extended from integer points to rational points

by

$$\Psi(x, y) = \frac{1}{k} \Psi(kx, ky), \quad (2.2.9)$$

where k is defined at the beginning of Step 2.

From (2.2.7), we get

$$\left(\frac{1}{n} T(z_1(n)) - \frac{1}{k} \Psi(kx, ky) \right) - \frac{1}{n} \max_{\pi \in \Pi[z_1(n), z_2(n)]} \sum_{z \in \pi} |X(z)| \leq \frac{1}{n} T(\lfloor nx \rfloor, \lfloor ny \rfloor) - \frac{1}{k} \Psi(kx, ky).$$

Similarly,

$$\frac{1}{n} T(\lfloor nx \rfloor, \lfloor ny \rfloor) - \frac{1}{k} \Psi(kx, ky) \leq \left(\frac{1}{n} T(z_2(n)) - \frac{1}{k} \Psi(kx, ky) \right) + \frac{1}{n} \max_{\pi \in \Pi[z_1(n), z_2(n)]} \sum_{z \in \pi} |X(z)|.$$

Note that for $i = 1, 2$, because $(kx, ky) \in \mathbb{N}^2$,

$$\lim_{n \rightarrow \infty} \mathbf{E} \left| \frac{1}{n} T(z_i(n)) - \frac{1}{k} \Psi(kx, ky) \right| = 0.$$

In addition,

$$\frac{1}{n} \mathbf{E} \left(\max_{\pi \in \Pi[z_1(n), z_2(n)]} \sum_{z \in \pi} |X(z)| \right) \leq \frac{1}{n} (kx + 1)(ky + 1) \mathbf{E} |X(0, 0)|,$$

which also converges to 0 as n goes to infinity.

Therefore the $L^1(\mathbf{P})$ convergence follows

$$\lim_{n \rightarrow \infty} \mathbf{E} \left| \frac{1}{n} T(\lfloor nx \rfloor, \lfloor ny \rfloor) - \frac{1}{k} \Psi(kx, ky) \right| = 0.$$

One can fairly easily check the following properties: for positive rational pairs (x_1, y_1)

and (x_2, y_2)

1. homogeneity: $\Psi(cx_1, cy_1) = c\Psi(x_1, y_1)$ for any positive rational number c .
2. superadditivity: $\Psi(x_1 + x_2, y_1 + y_2) \geq \Psi(x_1, y_1) + \Psi(x_2, y_2)$.
3. The above two together imply concavity: for rational $0 < c < 1$ and let $x = cx_1 + (1 - c)x_2$, $y = cy_1 + (1 - c)y_2$, we have

$$\Psi(x, y) \geq c\Psi(x_1, y_1) + (1 - c)\Psi(x_2, y_2). \quad (2.2.10)$$

Step 3: we extend the definition to $(x, y) \in (0, \infty)^2$. First, we prove $\lim_n \frac{T(n, \lfloor ny \rfloor)}{n}$ exists $\mathbf{P} - a.s.$ for all $y \in (0, \infty)$

If y is not rational, we can pick $y_1, y_2 \in \mathbb{Q} \cap (0, \infty)$ such that $y_1 < y < y_2$. By picking the optimal path from the origin to $(n, \lfloor ny_1 \rfloor)$ and then moving directly to $(n, \lfloor ny \rfloor)$, we have the following inequality:

$$T(n, \lfloor ny_1 \rfloor) + \sum_{j=\lfloor ny_1 \rfloor}^{\lfloor ny \rfloor - 1} X(n, j) \leq T(n, \lfloor ny \rfloor). \quad (2.2.11)$$

We now take a random subsequence $\{n_k, k = 1, 2, \dots\}$ such that

$$\frac{1}{n_k} T(n_k, \lfloor n_k y \rfloor) \rightarrow \liminf \frac{1}{n} T(n, \lfloor ny \rfloor).$$

By the strong law of large numbers $\frac{1}{n} \sum_{j=1}^{\lfloor ny \rfloor - \lfloor ny_1 \rfloor} X(0, j)$ converges to $(y - y_1)\mathbf{E}X(0, 0)$ $\mathbf{P} - a.s.$ and in probability. Therefore if we fix an $\varepsilon > 0$, then for every $\ell = 1, 2, \dots$ we

can find an integer $N(\ell)$ such that

$$\mathbf{P}\left(\left|\frac{1}{n} \sum_{j=1}^{\lfloor ny \rfloor - \lfloor ny_1 \rfloor} X(0, j) - (y - y_1) \mathbf{E}X(0, 0)\right| > \varepsilon\right) < 2^{-\ell} \quad (2.2.12)$$

for all $n > N(\ell)$.

Since $\frac{1}{n} \sum_{j=\lfloor ny_1 \rfloor}^{\lfloor ny \rfloor - 1} X(n, j)$ has the same distribution as $\frac{1}{n} \sum_{j=1}^{\lfloor ny \rfloor - \lfloor ny_1 \rfloor} X(0, j)$, (2.2.12) implies that from $\{n_k\}$ we can select a further subsequence $\{n_{k(l)}\}$ such that

$$\sum_{l=1}^{\infty} \mathbf{P}\left(\left|\frac{1}{n_{k(l)}} \sum_{j=\lfloor n_{k(l)} y_1 \rfloor}^{\lfloor n_{k(l)} y \rfloor - 1} X(n, j) - (y - y_1) \mathbf{E}X(0, 0)\right| > \varepsilon\right) < \infty,$$

and this shows that $\mathbf{P} - a.s.$ we have

$$\lim_{l \rightarrow \infty} \frac{1}{n_{k(l)}} \sum_{j=\lfloor n_{k(l)} y_1 \rfloor}^{\lfloor n_{k(l)} y \rfloor - 1} X(n, j) = (y - y_1) \mathbf{E}X(0, 0).$$

Hence by dividing through by n and taking limits along this subsequence $\{n_{k(l)}\}$ in (2.2.11) we get

$$\Psi(1, y_1) + (y - y_1) \mathbf{E}X(0, 0) \leq \liminf \frac{1}{n} T(n, \lfloor ny \rfloor). \quad (2.2.13)$$

Similarly, we can show

$$\limsup \frac{1}{n} T(n, \lfloor ny \rfloor) \leq \Psi(1, y_2) - (y_2 - y) \mathbf{E}X(0, 0). \quad (2.2.14)$$

Note that both of the above inequalities hold $\mathbf{P} - a.s.$

So now it is natural that we want to let y_1 and y_2 approach y from both sides. We

need the following lemma.

Lemma 2. *If $f(x)$ is a concave function defined on $\mathbb{Q} \cap (0, \infty)$, then it can be extended uniquely to a continuous function on $(0, \infty)$ by*

$$f(x) = \lim_{y \in \mathbb{Q}, y \rightarrow x} f(y). \quad (2.2.15)$$

Proof. Let $x > 0$ be a fixed real number. We first prove the one-sided limit $\lim_{y \in \mathbb{Q}, y \rightarrow x^-} f(y)$ is well-defined. If this is not the case, then we can find two sequences of rational numbers $\{u_n\}$ and $\{v_n\}$ such that they both approach x from below, and the limits $A_1 \equiv \lim_n f(u_n)$ and $A_2 \equiv \lim_n f(v_n)$ both exist with $A_1 > A_2$. Take $\varepsilon > 0$ small enough. We can find three rational numbers $u, u' \in \{u_n\}$ and $v \in \{v_n\}$ such that $u < v < u'$, and $f(u), f(u') > A_1 - \varepsilon$, $f(v) < A_2 + \varepsilon$. Take a rational number $0 < q < 1$ such that $v = qu + (1 - q)u'$, then

$$f(v) < A_2 + \varepsilon < A_1 - \varepsilon < qf(u) + (1 - q)f(u').$$

This contradicts (2.2.10), so the left limit $A = \lim_{y \in \mathbb{Q}, y \rightarrow x^-} f(y)$ exists. Similarly, $B = \lim_{y \in \mathbb{Q}, y \rightarrow x^+} f(y)$ also exists. We only need to show $A = B$.

Assume $A > B$, and choose $0 < \varepsilon < \frac{A-B}{2}$. Then there exists $\delta > 0$ such that for any rational numbers u and v with $x - \delta < u < x$ and $x < v < x + \delta$,

$$A - \varepsilon < f(u) < A + \varepsilon, \quad B - \varepsilon < f(v) < B + \varepsilon.$$

Take rational numbers u and v such that $x - \delta < u < x < v < x + \delta$ and $\frac{x-u}{v-u} < 1 - \frac{2\varepsilon}{A-B}$,

and pick a rational number $\alpha \in (\frac{x-u}{v-u}, 1 - \frac{2\varepsilon}{A-B})$. It follows that

$$\alpha f(v) + (1 - \alpha)f(u) > \alpha(B - \varepsilon) + (1 - \alpha)(A - \varepsilon) = \alpha B + (1 - \alpha)A - \varepsilon > B + \varepsilon.$$

However, by the choice of α , $\alpha v + (1 - \alpha)u$ is a rational number in (x, v) , so

$$f(\alpha v + (1 - \alpha)u) < B + \varepsilon < \alpha f(v) + (1 - \alpha)f(u).$$

This again violates (2.2.10), and by contradiction we reject $A > B$. Similarly, we can show $A < B$ is not possible either. Hence $A = B$ and $\lim_{y \in \mathbb{Q}, y \rightarrow x} f(y)$ exists.

For $x \in \mathbb{Q} \cap (0, \infty)$, we need to show this limit is consistent with the original value $f(x)$. We can repeat the above proof by contradiction and modify it when necessary. Specifically, we let $A = f(x)$, $B = \lim_{y \in \mathbb{Q}, y \rightarrow x^+} f(y)$ and assume $A > B$; in the following part we choose $u = x$ and $v \in (x, x + \delta)$ such that $f(v) \in (B - \varepsilon, B + \varepsilon)$, and take a rational number $\alpha \in (0, 1 - \frac{2\varepsilon}{A-B})$. We can check $f(\alpha v + (1 - \alpha)u) < \alpha f(v) + (1 - \alpha)f(u)$, so it contradicts (2.2.10) and $A > B$ is rejected. Similarly we reject $A < B$ and get $A = B$. Since the two-sided limit $\lim_{y \in \mathbb{Q}, y \rightarrow x} f(y)$ exists, this gives (2.2.15) for $x \in \mathbb{Q} \cap (0, \infty)$.

One can quickly check that the extension keeps concavity: for x and y in $(0, \infty)$ and $0 < c < 1$, take rational sequences $x_n \rightarrow x$, $y_n \rightarrow y$ and $c_n \rightarrow c$, then

$$\begin{aligned} f(cf(x) + (1 - c)f(y)) &= \lim_n f(c_n f(x_n) + (1 - c_n)f(y_n)) \\ &\geq \lim_n c_n f(x_n) + \lim_n (1 - c_n)f(y_n) \\ &= cf(x) + (1 - c)f(y). \end{aligned}$$

Since a finite concave function on an open set is continuous by Theorem 10.1 of [15], we get continuity. The uniqueness is trivial because any continuous function must satisfy (2.2.15). \square

Let us return to the proof of Proposition 2.1. Since $\Psi(1, y)$ is concave function defined on $\mathbb{Q} \cap (0, \infty)$, it can be extended to $(0, \infty)$ by (2.2.15). We let y_1 and y_2 approach y in (2.2.13) and (2.2.14), and get

$$\lim_{n \rightarrow \infty} \frac{1}{n} T(n, \lfloor ny \rfloor) = \lim_{u \in \mathbb{Q}, u \rightarrow y} \Psi(1, u) \quad \mathbf{P} - a.s.$$

Therefore we can extend the definition to $y \in (0, \infty)$

$$\Psi(1, y) = \lim_{u \in \mathbb{Q}, u \rightarrow y} \Psi(1, u). \quad (2.2.16)$$

We then extend the definition of $\Psi(x, y)$ to any $(x, y) \in (0, \infty)^2$, and show that

$$\lim_{n \rightarrow \infty} \frac{1}{n} T(\lfloor nx \rfloor, \lfloor ny \rfloor) = x \Psi(1, \frac{y}{x}) \quad \mathbf{P} - a.s. \quad (2.2.17)$$

If we write $m = \lfloor nx \rfloor$, then $m \leq nx < m + 1$, hence $\lfloor m \frac{y}{x} \rfloor \leq \lfloor ny \rfloor \leq \lfloor (m + 1) \frac{y}{x} \rfloor$. It is clear that

$$\begin{aligned} T(m, \lfloor m \frac{y}{x} \rfloor) - \sum_{i=\lfloor m \frac{y}{x} \rfloor}^{\lfloor ny \rfloor - 1} |X(m, i)| &\leq T(\lfloor nx \rfloor, \lfloor ny \rfloor) \\ &\leq T(m, \lfloor (m + 1) \frac{y}{x} \rfloor) + \sum_{i=\lfloor ny \rfloor}^{\lfloor (m + 1) \frac{y}{x} \rfloor - 1} |X(m, i)|. \end{aligned} \quad (2.2.18)$$

Similarly to (2.2.8), we can run a Borel-Cantelli argument and claim that as n goes

to infinity,

$$\frac{1}{n} \sum_{i=\lfloor m \frac{y}{x} \rfloor}^{\lfloor ny \rfloor} |X(m, i)| \rightarrow 0 \quad \text{and} \quad \frac{1}{n} \sum_{i=\lfloor ny \rfloor}^{\lfloor (m+1) \frac{y}{x} \rfloor} |X(m, i)| \rightarrow 0 \quad \mathbf{P} - a.s.$$

Therefore dividing through by n and taking limits in (2.2.18) gives that $\mathbf{P} - a.s.$

$$x\Psi(1, \frac{y}{x}) \leq \liminf_{n \rightarrow \infty} \frac{1}{n} T(\lfloor nx \rfloor, \lfloor ny \rfloor) \leq \limsup_{n \rightarrow \infty} \frac{1}{n} T(\lfloor nx \rfloor, \lfloor ny \rfloor) \leq x\Psi(1, \frac{y}{x}).$$

So (2.2.17) is proved. We can then define for $(x, y) \in (0, \infty)^2$ that

$$\Psi(x, y) = x\Psi(1, \frac{y}{x}). \quad (2.2.19)$$

Finally, we prove $L^1(\mathbf{P})$ convergence for $(x, y) \in (0, \infty)^2$. From (2.2.11) and its counterpart in the other direction, we get

$$\begin{aligned} & \left(\frac{1}{n} T(n, \lfloor ny_1 \rfloor) - \Psi(1, y_1) \right) + \left(\Psi(1, y_1) - \Psi(1, y) \right) + \frac{1}{n} \sum_{j=\lfloor ny_1 \rfloor}^{\lfloor ny \rfloor - 1} X(n, j) \\ & \leq \frac{1}{n} T(n, \lfloor ny \rfloor) - \Psi(1, y) \\ & \leq \left(\frac{1}{n} T(n, \lfloor ny_2 \rfloor) - \Psi(1, y_2) \right) + \left(\Psi(1, y_2) - \Psi(1, y) \right) + \frac{1}{n} \sum_{j=\lfloor ny \rfloor}^{\lfloor ny_2 \rfloor - 1} X(n, j). \end{aligned}$$

We have shown that for rational numbers y_1 and y_2 , $\lim_{n \rightarrow \infty} \mathbf{E} \left| \frac{1}{n} T(n, \lfloor ny_i \rfloor) - \Psi(1, y_i) \right| =$

0. Then

$$\begin{aligned}
& \lim_{n \rightarrow \infty} \mathbf{E} \left| \frac{1}{n} T(n, \lfloor ny \rfloor) - \Psi(1, y) \right| \\
& \leq \left(|\Psi(1, y_2) - \Psi(1, y)| + (y_2 - y) \mathbf{E} |X(0, 0)| \right) \\
& \quad \vee \left(|\Psi(1, y_1) - \Psi(1, y)| + (y - y_1) \mathbf{E} |X(0, 0)| \right)
\end{aligned}$$

We let y_1 and y_2 approach y and get

$$\lim_{n \rightarrow \infty} \mathbf{E} \left| \frac{1}{n} T(n, \lfloor ny \rfloor) - \Psi(1, y) \right| = 0.$$

We can use a very similar argument starting from (2.2.18) to get

$$\lim_{n \rightarrow \infty} \mathbf{E} \left| \frac{1}{n} T(\lfloor nx \rfloor, \lfloor ny \rfloor) - x \Psi(1, \frac{y}{x}) \right| = 0.$$

So $L^1(\mathbf{P})$ convergence is proved.

So now we have started from the definition of $\Psi(x, y)$ on integer points and extended it to $(0, \infty)^2$ by (2.2.9), (2.2.16) and (2.2.19). We can immediately extend the homogeneity, superadditivity, and concavity conditions to real points. Again by Theorem 10.1 of [15] a finite concave function on an open set is continuous, we get continuity.

Now we have finished the proof of Proposition 2.1. □

We may also define $\Psi(x, y)$ using supremum. If we denote $x_n = \lfloor nx \rfloor$ and $y_n = \lfloor ny \rfloor$,

by superadditivity, we have

$$\begin{aligned} T(x_m, y_m) + T((x_m, y_m), (x_m + x_n, y_m + y_n)) \\ + T((x_m + x_n, y_m + y_n), (x_{m+n}, y_{m+n})) \leq T(x_{m+n}, y_{m+n}). \end{aligned}$$

We note that $x_{m+n} - x_m - x_n = 0$ or 1 , and so is $y_{m+n} - y_m - y_n$. If $\mathbf{E}X(0, 0) \geq 0$, then we can easily check that $\mathbf{E}T(0, 1)$, $\mathbf{E}T(1, 0)$ and $\mathbf{E}T(1, 1)$ are nonnegative. This gives $\mathbf{E}T(x_m, y_m) + \mathbf{E}T(x_n, y_n) \leq \mathbf{E}T(x_{m+n}, y_{m+n})$, which leads to another definition that

$$\Psi(x, y) = \lim_{n \rightarrow \infty} \frac{1}{n} \mathbf{E}T(x_n, y_n) = \sup_n \frac{1}{n} \mathbf{E}T(x_n, y_n) = \sup_n \frac{1}{n} \mathbf{E}T(\lfloor nx \rfloor, \lfloor ny \rfloor).$$

This alternative definition will be helpful in some settings where we need to estimate the upper bound of last-passage times. It may not be true if $\mathbf{E}X(0, 0) < 0$. An easy counterexample is the case where $X(z) \equiv -1$ for all $z \in \mathbb{Z}_+^2$. We easily see that $\frac{1}{n} \mathbf{E}T(\lfloor nx \rfloor, \lfloor ny \rfloor) = \frac{1}{n}(-\lfloor nx \rfloor - \lfloor ny \rfloor) > \frac{1}{n}(-nx - ny) = -x - y = \Psi(x, y)$ when x and y are not integers. However, if x and y are both integers, $\Psi(x, y) = \sup_n \frac{1}{n} \mathbf{E}T(nx, ny)$ is a valid definition regardless of the sign of $\mathbf{E}X(0, 0)$.

Chapter 3

Results for Bernoulli and exponential models

3.1 Bernoulli models with strict-weak paths in a random environment

As we have seen in the proof of Proposition 2.1, last-passage models with Bernoulli-distributed weights can play an important role when we study general models. As for models with Bernoulli weights, one of the major difficulties is that there is no explicit results so far about $\Psi(x, y)$ with the weakly increasing paths. For this reason in this chapter we first study Bernoulli models with two different types of admissible paths, and eventually give an estimate of $\Psi(x, y)$ with the weakly increasing paths.

The environment is now an i.i.d. sequence $\{p_j\}_{j \in \mathbb{Z}_+}$ of numbers $p_j \in [0, 1]$, with distribution \mathbb{P} . Given $\{p_j\}$, the weights $\{X(i, j)\}$ are independent with marginal distributions $P(X(i, j) = 1) = p_j = 1 - P(X(i, j) = 0)$. We consider two last-passage times that differ by the type of admissible path: for $z_1, z_2 \in \mathbb{Z}_+^2$

$$T_{\rightarrow}(z_1, z_2) = \max_{\pi \in \Pi_{\rightarrow}[z_1, z_2]} \sum_{z \in \pi} X(z) \quad \text{and} \quad T_{\uparrow}(z_1, z_2) = \max_{\pi \in \Pi_{\uparrow}[z_1, z_2]} \sum_{z \in \pi} X(z). \quad (3.1.1)$$

In terms of coordinates denote the endpoints by $z_k = (a_k, b_k)$, $k = 1, 2$. Admissible paths $\pi \in \Pi_{\rightarrow}[z_1, z_2)$ are of the form $\pi = \{(a_1, y_0)(a_1 + 1, y_1), \dots, (a_2 - 1, y_{a_2 - a_1 - 1})\}$ with $b_1 \leq y_0 \leq y_1 \leq \dots \leq y_{a_2 - a_1 - 1} \leq b_2$. Please see Figure 2. Again note that now we always exclude the end point as was declared in Remark 1.

Symmetrically paths $\pi \in \Pi_{\uparrow}[z_1, z_2)$ are of the form $\pi = \{(x_0, b_1), (x_1, b_1 + 1) \dots, (x_{b_2 - b_1 - 1}, b_2 - 1)\}$ with $a_1 \leq x_0 \leq x_1 \leq \dots \leq x_{b_2 - b_1 - 1} \leq a_2$. Thus paths in $\Pi_{\rightarrow}[z_1, z_2)$ increase strictly in the x -direction while those in $\Pi_{\uparrow}[z_1, z_2)$ increase strictly in the y -direction.

As before we simplify notation with $T_{\rightarrow}(0, z) = T_{\rightarrow}(z)$. The almost sure limits are denoted by

$$\Psi_{\rightarrow}(x, y) = \lim_{n \rightarrow \infty} \frac{1}{n} T_{\rightarrow}(\lfloor nx \rfloor, \lfloor ny \rfloor) \quad \text{and} \quad \Psi_{\uparrow}(x, y) = \lim_{n \rightarrow \infty} \frac{1}{n} T_{\uparrow}(\lfloor nx \rfloor, \lfloor ny \rfloor) \quad (3.1.2)$$

for $(x, y) \in (0, \infty)^2$. The proof of the existence of the above limits can be outlined as follows: we first verify the assumptions of the subadditive ergodic theorem for $(x, y) \in \mathbb{N}^2$, and note that the moment assumption is trivial for Bernoulli models because of the uniform boundedness; then we extend the definition to all $(x, y) \in (0, \infty)^2$ using the same argument as we had in the previous chapter. We will omit the details and claim the existence of the limits.

Remark 3. In (2.2.6) we used a result (3.1.8) from this chapter. To remove the concern for circularity, we note that the proof of Proposition 2.1 works for the Bernoulli models even without knowing (3.1.8). The logic progression actually should be: Proposition 2.1 holds for Bernoulli models, then we derive (3.1.8), and apply (3.1.8) to prove Proposition 2.1 for more general models under moment conditions.

The next theorem gives the explicit limits. (3.1.3) is the same as in [9, Thm. 1]. Inside the $\mathbb{E}[\dots]$ expectations below p is the random Bernoulli probability. Let $b = \operatorname{ess\,sup}_{\mathbb{P}} p$ denote the maximal probability.

We prove the formulas and inequalities first for Ψ_{\rightarrow} and then for Ψ_{\uparrow} . It is convenient to assume $b < 1$. Results for the case $b = 1$ follow by taking a limit.

Theorem 3.1.1. *The limits in (3.1.2) are as follows for $x, y \in (0, \infty)$.*

$$\Psi_{\rightarrow}(x, y) = \begin{cases} bx + y(1 - b)\mathbb{E}\left[\frac{p}{b-p}\right], & x/y \geq \mathbb{E}\left[\frac{p(1-p)}{(b-p)^2}\right] \\ yz_0^2\mathbb{E}\left[\frac{1-p}{(z_0-p)^2}\right] - y, & \mathbb{E}\left[\frac{p}{1-p}\right] < x/y < \mathbb{E}\left[\frac{p(1-p)}{(b-p)^2}\right] \\ x, & 0 < x/y \leq \mathbb{E}\left[\frac{p}{1-p}\right] \end{cases} \quad (3.1.3)$$

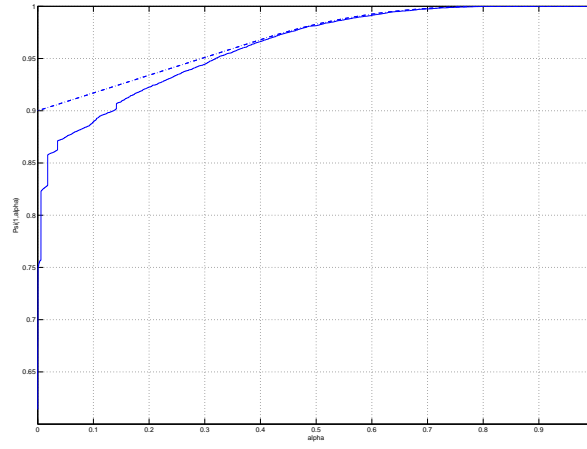
with $z_0 \in (b, 1)$ uniquely defined by the equation

$$x/y = \mathbb{E}\left[\frac{p(1-p)}{(z_0-p)^2}\right].$$

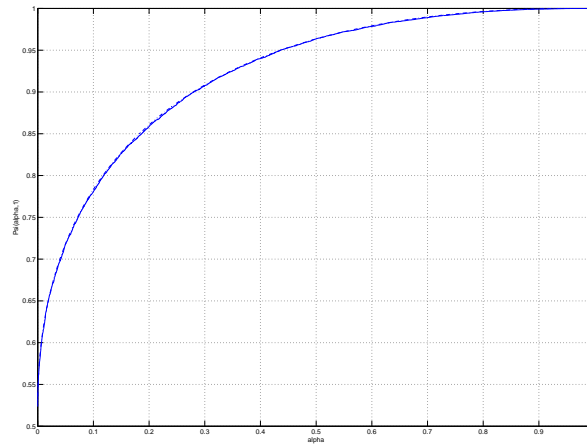
$$\Psi_{\uparrow}(x, y) = \begin{cases} y - yz_0^2\mathbb{E}\left[\frac{1-p}{(z_0+p)^2}\right], & 0 < x/y < \mathbb{E}\left[\frac{1-p}{p}\right] \\ y, & x/y \geq \mathbb{E}\left[\frac{1-p}{p}\right] \end{cases} \quad (3.1.4)$$

with $z_0 \in (0, \infty)$ uniquely defined by the equation

$$x/y = \mathbb{E}\left[\frac{p(1-p)}{(z_0+p)^2}\right].$$



(a) Approximation of the function $\Psi_{\rightarrow}(1, \alpha)$ from simulation.



(b) Approximation of the function $\Psi_{\uparrow}(\alpha, 1)$ from simulation.

Figure 3: simulations to verify Theorem 3.1.1

To illustrate this result, we present two plots based on simulations: the rates p_j are chosen to be i.i.d. with $\mathbb{P}(p_j \leq x) = 1 - (\frac{0.9-x}{0.5})^3$ supported on $[0.4, 0.9]$.

The dashed curves are the precise values described by Theorem 3.1.1. The solid curves are the approximations $\frac{T_{\rightarrow}(n, \lfloor n\alpha \rfloor)}{n}$ and $\frac{T_{\uparrow}(\lfloor n\alpha \rfloor, n)}{n}$ respectively when $n = 20,000$.

In Figure 3.3(a), the approximation is not very accurate when α is close to 0 after zooming in. The reason is that the density of Bernoulli rates near b is low, so the first few rows are not likely to have means close to b . In Figure 3.3(b), the error is almost negligible. We can also clearly see that $\Psi_{\rightarrow}(1, \alpha)$ and $\Psi_{\uparrow}(\alpha, 1)$ approach to different limits as $\alpha \searrow 0$.

Our second result gives simplified bounds that are useful for the proof of the Theorem 4.1.1 in the next chapter. Let $\bar{p} = \mathbb{E}(p)$ be the mean of the environment. $\Psi(x, y)$ is the limiting time constant with weakly increasing paths defined in Proposition 2.1.

Theorem 3.1.2. *The following three inequalities hold for the Bernoulli model:*

$$\Psi_{\rightarrow}(x, y) \leq bx + 2\sqrt{\bar{p}(1-b)xy}, \quad (3.1.5)$$

$$\Psi_{\uparrow}(x, y) \leq \bar{p}y + 2\sqrt{\bar{p}(1-\bar{p})xy} \quad (3.1.6)$$

and

$$\Psi(x, y) \leq \bar{p}y + 4\sqrt{\bar{p}(1-\bar{p})xy} + bx. \quad (3.1.7)$$

(3.1.7) follows from (3.1.5) and (3.1.6) because $\Psi(x, y) \leq \Psi_{\rightarrow}(x, y) + \Psi_{\uparrow}(x, y)$. Another loose estimate we will use later following (3.1.7) is

$$\Psi(x, y) \leq \bar{p}y + 4\sqrt{\bar{p}(1-\bar{p})xy} + bx \leq (y + 4\sqrt{xy})\sqrt{\bar{p}} + bx. \quad (3.1.8)$$

Proof of (3.1.3) and (3.1.5). We adapt the proof from [19] to the random environment situation and sketch the main points. Results from [19] are directly applicable in the random environment and will be quoted without further explanation.

Consider now the environment $\{p_j\}$ fixed, but the weights $X(i, j)$ random. For integers $0 \leq s < t$ and a, k define an inverse to the last-passage time as

$$\Gamma((a, s), k, t) = \min\{l \in \mathbb{Z}_+ : T_{\rightarrow}((a+1, s+1), (a+l+1, t)) \geq k\}.$$

For the special case $k = 0$ we define $\Gamma((a, s), 0, t) = 0$, but $\Gamma((a, s), k, t) > 0$ for $k > 0$. Knowing the limits of the variables Γ is the same as knowing Ψ_{\rightarrow} . By the homogeneity of Ψ_{\rightarrow} it is enough to find $h(x) = \Psi_{\rightarrow}(x, 1)$. By the homogeneity and superadditivity of Ψ_{\rightarrow} , h is concave and nondecreasing. Let g be the inverse function of h on \mathbb{R}_+ . Then g is convex and nondecreasing, and (7.4) in [19] still holds here:

$$tg(x/t) = \lim_{n \rightarrow \infty} \frac{1}{n} \Gamma((0, 0), \lfloor nx \rfloor, \lfloor nt \rfloor).$$

To find these functions we construct an exclusion-type process $z(t) = \{z_k(t) : k \in \mathbb{Z}\}$ of labeled, ordered particles $z_k(t) < z_{k+1}(t)$ that jump leftward on the lattice \mathbb{Z} , in discrete time $t \in \mathbb{Z}_+$. Given an initial configuration $\{z_i(0)\}$ that satisfies $z_{i-1}(0) \leq$

$z_i(0) - 1$ and $\liminf_{i \rightarrow -\infty} |i|^{-1} z_i(0) > -1/b$, the evolution is defined by

$$z_k(t) = \inf_{i: i \leq k} \{z_i(0) + \Gamma((z_i(0), 0), k - i, t)\}, \quad k \in \mathbb{Z}, t \in \mathbb{N}. \quad (3.1.9)$$

It can be checked that $z(t)$ is a well-defined Markov process, in particular that $z_k(t) > -\infty$ almost surely. These claims are identical to Lemma 5.2 and Lemma 5.3 in [19].

Define the process $\{\eta_i(t)\}$ of interparticle distances by $\eta_i(t) = z_{i+1}(t) - z_i(t)$ for $i \in \mathbb{Z}$ and $t \in \mathbb{Z}_+$. By Prop. 1 in [19] process $\{\eta_i(t)\}$ has a family of i.i.d. geometric invariant distributions indexed by the mean $u \in [1, b^{-1})$ and defined by

$$P(\eta_i = n) = u^{-1}(1 - u^{-1})^{n-1}, \quad n \in \mathbb{N}. \quad (3.1.10)$$

Let $x_k(t) = z_k(t) - z_k(t-1) \geq 0$ be the absolute size of the jump of the k th particle from time $t-1$ to t , and let $q_t = 1 - p_t$. From (6.5) in [19], in the stationary process

$$P(x_k(t) = x) = \begin{cases} (1 - up_t)q_t^{-1} & x = 0 \\ p_t(1 - up_t)q_t^{-1}(u - 1)^x(uq_t)^{-x} & x = 1, 2, 3, \dots \end{cases} \quad (3.1.11)$$

We track the motion of particle $z_0(t)$ in a stationary situation. The initial state is defined by setting $z_0(0) = 0$ and by letting $\{\eta_i(0)\}$ be i.i.d. with common distribution (3.1.10). With $k = 0$, divide by t in (3.1.9) and take $t \rightarrow \infty$. Apply laws of large numbers inside the braces in (3.1.9), with some simple estimation to pass the limit through the infimum, to find the average speed of the tagged particle:

$$- \lim_{t \rightarrow \infty} \frac{1}{t} z_0(t) = \sup_{x \geq 0} \{ux - g(x)\} \equiv f(u). \quad (3.1.12)$$

For further details please refer to the proof of (7.15) in [19].

The last equality defines the speed f as $f = g^+$, the *monotone conjugate* of g . It is natural to set $f(u) = 0$ for $u \in [0, 1)$, $f(b^{-1}) = f((b^{-1})-)$, and $f(u) = \infty$ for $u > b^{-1}$. By [16, Thm. 12.4]

$$g(x) = \sup_{u \geq 0} \{xu - g^+(u)\} = \sup_{1 \leq u \leq 1/b} \{xu - f(u)\}. \quad (3.1.13)$$

Since $z_0(t)$ is a sum of jumps $x_0(k)$ with distribution (3.1.11) we have the second moment bound $\sup_{t \in \mathbb{N}} \mathbf{E}[(t^{-1}z_0(t))^2] < \infty$, and consequently the limit in (3.1.12) holds also in expectation. From this

$$\begin{aligned} f(u) &= - \lim_{t \rightarrow \infty} \mathbf{E}[t^{-1}z_0(t)] = \lim_{t \rightarrow \infty} \mathbf{E}\left[t^{-1} \sum_{k=1}^t x_0(k)\right] = \mathbf{E}[x_0(0)] \\ &= \mathbf{E} \sum_{x=1}^{\infty} x(u-1)^x (uq)^{-x} p(1-up)(1-p)^{-1} = \mathbf{E}\left[\frac{pu(u-1)}{1-up}\right]. \end{aligned} \quad (3.1.14)$$

Next we will find the explicit expression of $g(x)$ from (3.1.14) and (3.1.13). To find the supremum of $xu - f(x)$, we compute its first derivative and find it equal to $x - \mathbf{E}\left[\frac{1-p}{(b-p)^2} - 1\right]$.

When $\mathbf{E}\left[\frac{p}{1-p}\right] \leq x \leq \mathbf{E}\left[\frac{1-p}{(b-p)^2} - 1\right]$, the equation

$$x + 1 = \mathbf{E}\left[\frac{1-p}{(1-u_0p)^2}\right] \quad (3.1.15)$$

has a solution $u_0 \in [1, \frac{1}{b}]$, so $g(x) = xu_0 - f(u_0)$. If $x < \mathbf{E}\left[\frac{p}{1-p}\right]$, then $g(x) = x$. If

$x > \mathbb{E}\left[\frac{1-p}{(b-p)^2} - 1\right]$, $g(x) = \frac{x}{b} - f(\frac{1}{b})$. Therefore,

$$g(x) = \begin{cases} x/b - b^{-1}(1-b)\mathbb{E}\left[\frac{p}{(b-p)}\right] & x \geq b^2\mathbb{E}\left[\frac{(1-p)}{(b-p)^2} - 1\right] \\ u_0^2\mathbb{E}\left[\frac{p(1-p)}{(1-u_0p)^2}\right] & \mathbb{E}\left[\frac{p}{1-p}\right] < x < b^2\mathbb{E}\left[\frac{(1-p)}{(b-p)^2} - 1\right] \\ x & 0 < x \leq \mathbb{E}\left[\frac{p}{1-p}\right] \end{cases} \quad (3.1.16)$$

where $u_0 \in (1, b^{-1})$ is uniquely defined by the equation

$$x + 1 = \mathbb{E}\left[(1-p)(1-u_0p)^{-2}\right].$$

Then we need to find the inverse function $h(x) = g^{-1}(x)$ and then $\Psi_{\rightarrow}(x, y) = yh(x/y)$. $g(x)$ has three different cases when x takes different values. The first and last cases $g^{-1}(x)$ can be calculated directly, and for the second case we only need to interchange the positions of x and $g(x)$ in their defining equations. Therefore

$$h(x) = g^{-1}(x) = \begin{cases} bx + \mathbb{E}\left[\frac{(1-b)p}{b-p}\right] & x > \mathbb{E}\left[\frac{p(1-p)}{(b-p)^2}\right] \\ \mathbb{E}\left[\frac{(1-p)}{(1-u_0p)^2} - 1\right] & \mathbb{E}\left[\frac{p}{1-p}\right] \leq x \leq \mathbb{E}\left[\frac{p(1-p)}{(b-p)^2}\right] \\ x & 0 \leq x < \mathbb{E}\left[\frac{p}{1-p}\right] \end{cases} \quad (3.1.17)$$

with

$$x = \mathbb{E}\left[\frac{u_0^2 p(1-p)}{(1-u_0p)^2}\right].$$

Since $\Psi_{\rightarrow}(x, y) = yh(x/y)$, (3.1.3) proved.

To prove (3.1.5) we return to the duality (3.1.13) and write

$$g(x) \geq \sup_{1 \leq u < 1/b} \{xu - \tilde{f}(u)\} \quad \text{for} \quad \tilde{f}(u) = \frac{u(u-1)}{1-ub} \bar{p}. \quad (3.1.18)$$

$\tilde{f}'(u) = x$ is solved by $u^* = b^{-1} \left(1 - \sqrt{\frac{(1-b)\bar{p}}{bx+\bar{p}}} \right)$.

When $x \geq \frac{\bar{p}}{1-b}$, we have $u^* \in [1, \frac{1}{b})$, and then

$$g(x) \geq xu^* - \tilde{f}(u^*) = \frac{1}{b^2} \left(\sqrt{(1-b)\bar{p}} - \sqrt{bx + \bar{p}} \right)^2.$$

Consequently

$$g^{-1}(x) \leq \frac{1}{b} \left(\sqrt{b^2 x} + \sqrt{(1-b)\bar{p}} \right)^2 - \frac{\bar{p}}{b} = bx - \bar{p} + 2\sqrt{(1-b)\bar{p}x}.$$

When $x < \frac{\bar{p}}{1-b}$, the supremum in (3.1.18) is attained at $u = 1$, and in this case

$$g^{-1}(x) \leq x \leq bx + 2\sqrt{(1-b)\bar{p}x}.$$

The bound (3.1.5) now follows from $\Psi_{\rightarrow}(x, y) = yg^{-1}(x/y)$. □

Proof of (3.1.4) and (3.1.6). The scheme is the same as above, so we omit some more details. The inverse of the last-passage time is now defined

$$\Gamma((a, s), k, t) = \min\{l \in \mathbb{Z}_+ : T_{\uparrow}((a, s+1), (a+l, t+1)) \geq k\}.$$

When $k = 0$ we define $\Gamma((a, s), 0, t) = 0$. Vertical distance $t - s$ allows for at most $t - s$ points with value 1, so the above quantity must be set equal to ∞ for $k > t - s$. The

particle process $\{z(t) : t \in \mathbb{Z}_+\}$ is defined by the same formula (3.1.9) as before but it is qualitatively different. The particles still jump to the left, but the ordering rule is now $z_k(t) \leq z_{k+1}(t)$ so particles are allowed to sit on top of each other. Well-definedness of the dynamics needs no further restrictions on admissible particle configurations because the minimum in (3.1.9) only considers $i \in \{k-t, \dots, k\}$ so it is well-defined for all initial configurations $\{z_i(0) : i \in \mathbb{Z}\}$ such that $z_i(0) \leq z_{i+1}(0)$.

The following can be checked. Under a fixed environment $\{p_j\}$, the gap process $\{\eta_i(t) = z_{i+1}(t) - z_i(t) : i \in \mathbb{Z}\}$ has i.i.d. geometric invariant distributions $P(\eta_k = n) = (\frac{1}{1+u})(\frac{u}{1+u})^n$, $n \in \mathbb{Z}_+$, indexed by the mean $u \in \mathbb{R}_+$. In this stationary situation the successive jumps $x_k(t) = z_k(t-1) - z_k(t)$ of a tagged particle have distribution

$$P(x_k(t) = y) = \begin{cases} \frac{1}{1+up_t} & y = 0 \\ (\frac{u}{u+1})^y \frac{p_t}{1+up_t} & y \geq 1. \end{cases}$$

From here the analysis proceeds the same way as for the other model. The speed function is defined by

$$f(u) = - \lim_{n \rightarrow \infty} \mathbf{E}[n^{-1}z_0(n)] = \mathbf{E}[x_0(0)] = u(u+1)\mathbb{E}\left[\frac{p}{1+up}\right].$$

We now calculate $g(x) = \sup_{u \geq 0} \{xu - f(u)\}$.

When $\bar{p} \leq x \leq 1$, there exists a non-negative solution u_0 to the equation

$$x = f'(x) = 1 - \mathbb{E}\left[\frac{1-p}{(1+up)^2}\right], \quad (3.1.19)$$

and it follows that

$$g(x) = xu_0 - f(u_0) = \mathbb{E} \left[\frac{u_0^2 p(1-p)}{(1+u_0 p)^2} \right].$$

When $x < \bar{p}$, we can check the sup is taken at $u = 0$ and thus $g(x) = 0$. If $x > 1$, $g(x) = +\infty$ so

$$g(x) = \begin{cases} +\infty & x > 1 \\ \mathbb{E} \left[\frac{u_0^2 p(1-p)}{(1+u_0 p)^2} \right] & \bar{p} \leq x \leq 1 \\ 0 & 0 \leq x \leq \bar{p} \end{cases} \quad (3.1.20)$$

and it has an inverse function

$$g^{-1}(x) = \begin{cases} 1 - \mathbb{E} \left[\frac{(1-p)}{(1+u_0 p)^2} \right] & 0 < x \leq \mathbb{E} \left[\frac{1-p}{p} \right] \\ 1 & x > \mathbb{E} \left[\frac{1-p}{p} \right] \end{cases}$$

with u_0 defined implicitly in

$$x = \mathbb{E} \left[\frac{u_0^2 p(1-p)}{(1+u_0 p)^2} \right].$$

Since $\Psi_{\uparrow}(x, y) = yg^{-1}(\frac{x}{y})$, (3.1.4) follows easily.

To prove (3.1.6), note that

$$\begin{aligned} g(x) &= \sup_{u \geq 0} \{xu - f(u)\} \geq \sup_{u \geq 0} \left\{ xu - \frac{\bar{p}u(u+1)}{1+u\bar{p}} \right\} \\ &= \begin{cases} \frac{1}{\bar{p}}(\sqrt{1-x} - \sqrt{1-\bar{p}})^2 & \bar{p} \leq x \leq 1 \\ 0 & 0 \leq x \leq \bar{p}. \end{cases} \end{aligned}$$

We used Jensen's inequality and concavity of $p \mapsto \frac{p}{1+up}$. From this

$$g^{-1}(x) \leq \begin{cases} \bar{p} - \bar{p}x + 2\sqrt{\bar{p}(1-\bar{p})x} & 0 \leq x \leq \frac{1-\bar{p}}{\bar{p}} \\ 1 & x > \frac{1-\bar{p}}{\bar{p}} \end{cases}$$

and (3.1.6) follows. \square

3.2 Limiting shapes for exponential models

3.2.1 Estimate of $\Psi_G(\alpha, 1)$

In this section, we consider both cases $\Psi(1, \alpha)$ and $\Psi(\alpha, 1)$ for the exponential model where some (partially) explicit calculation is possible.

Let $\{\xi_j\}_{j \in \mathbb{Z}_+}$ be an i.i.d. sequence of random variables $0 < c \leq \xi_j$ with common distribution m . We assume c is the exact lower bound: $m[c, c + \varepsilon) > 0$ for each $\varepsilon > 0$. The distribution function of exponential distribution with random parameter ξ_j is $G_j(x) = 1 - e^{-\xi_j x}$ for $x > 0$. Its mean is $\frac{1}{\xi_j}$ and the variance is $\frac{1}{\xi_j^2}$. Then the essential supremum of the random mean is $\mu^* = c^{-1}$.

We assume $X(i, j) \sim G_j(x)$ for $(i, j) \in \mathbb{Z}_+^2$ and write Ψ_G for the limiting time constant defined in (2.2.4). Define $\mu_G = \int_{[c, \infty)} \frac{1}{\xi} m(d\xi)$ and $\sigma_G^2 = \int_{[c, \infty)} \frac{1}{\xi^2} m(d\xi)$.

An implicit description of the limit shape was derived in [20] by way of studying an exclusion process with random jump rates attached to particles. We recall the result from [20] here. One explicit shape is needed for the proof of Theorem 4.1.1 also, so this result will serve there too.

Define first a critical value $u^* = \int_{[c, \infty)} \frac{c}{\xi - c} m(d\xi) \in (0, \infty]$. For $0 \leq u < u^*$ define

$a = a(u)$ implicitly by

$$u = \int_{[c, \infty)} \frac{a}{\xi - a} m(d\xi).$$

The function $a(u)$ is strictly increasing, strictly concave, continuously differentiable and one-to-one from $0 < u < u^*$ onto $0 < a < c$. We let $a(u) = c$ for $u \geq u^*$ if $u^* < \infty$.

Then define $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ by

$$g(y) = \sup_{u \geq 0} \{-yu + a(u)\}, \quad y \geq 0. \quad (3.2.1)$$

The function g is monotone decreasing, continuous, and $g(y) = 0$ for $y \geq a'(0+) = 1/\mu_G$. It is the level curve of the time constant. The equations connecting the two are $g(y) = \inf\{x > 0 : \Psi_G(x, y) \geq 1\}$ and

$$\Psi_G(x, y) = \inf\{t \geq 0 : tg(y/t) \geq x\}. \quad (3.2.2)$$

We first derive a result of the asymptotic behavior of $\Psi(x, y)$ close to the y -axis. From homogeneity, we can focus on the univariate function $\Psi(1, \alpha)$.

Theorem 3.2.1. *For the random exponential distributions defined above, as $\alpha \searrow 0$*

$$\Psi_G(\alpha, 1) = \mu_G + 2\sigma_G\sqrt{\alpha} + O(\alpha).$$

Proof. Recall the definition of the limit shape $\Psi_G(\alpha, 1)$ from (3.2.2). From (3.2.1) one can read that $tg(1/t)$ is nondecreasing in t . Thus by (3.2.2) $\Psi_G(\alpha, 1) = t = t(\alpha)$ such that $tg(1/t) = \alpha$.

Next we argue that when α is close enough to 0, $g(1/t) = -u_0/t + a(u_0)$ for some

$0 < u_0 < u^*$ with $a'(u_0) = 1/t$. Since $a(0) = 0$ and $a(u^*-) = c$, strict concavity gives for $0 < u < u^*$

$$\begin{aligned} \left\{ \int_{[c,\infty)} \frac{\xi}{(\xi - c)^2} m(d\xi) \right\}^{-1} &= a'(u^*-) < a'(u) = \left\{ \int_{[c,\infty)} \frac{\xi}{(\xi - a(u))^2} m(d\xi) \right\}^{-1} \\ &< a'(0+) = \left\{ \int_{[c,\infty)} \xi^{-1} m(d\xi) \right\}^{-1} = \frac{1}{\mu_G}. \end{aligned}$$

On the other hand, $0 < \Psi_G(\alpha, 1) - \mu_G \leq C\sqrt{\alpha} + C\alpha$ where the second inequality comes from comparing $\{G_j\}$ in (4.1.3) with identically zero weights. Thus when α is small enough, $1/t$ is in the range of a' . Consequently there exists $u_0 \in (0, u^*)$ such that $a'(u_0) = 1/t$, or equivalently,

$$\int_{[c,\infty)} \frac{\xi}{(\xi - a(u_0))^2} m(d\xi) = t. \quad (3.2.3)$$

From the choice of t , $\alpha = tg(1/t) = t(-u_0/t + a(u_0)) = -u_0 + ta(u_0)$ and so

$$\Psi_G(\alpha, 1) = t = \frac{\alpha}{a(u_0)} + \frac{u_0}{a(u_0)} = \frac{\alpha}{a(u_0)} + \int_{[c,\infty)} \frac{1}{\xi - a(u_0)} m(d\xi). \quad (3.2.4)$$

Combining (3.2.3) and (3.2.4) gives

$$\alpha = a(u_0)^2 \int_{[c,\infty)} \frac{1}{(\xi - a(u_0))^2} m(d\xi). \quad (3.2.5)$$

From this

$$a(u_0)^2 \sigma_G^2 = a(u_0)^2 \int_{[c,\infty)} \frac{1}{\xi^2} m(d\xi) \leq \alpha.$$

Hence we have $0 \leq a(u_0) \leq \sqrt{\alpha}/\sigma_G$.

When α and hence $a(u_0)$ is small, (3.2.5) and the last bound on $a(u_0)$ yield

$$\begin{aligned}
0 \leq \frac{\alpha}{a(u_0)^2} - \sigma_G^2 &= \int_{[c, \infty)} \left[\frac{1}{(\xi - a(u_0))^2} - \frac{1}{\xi^2} \right] m(d\xi) \\
&= \int_{[c, \infty)} \frac{2\xi a(u_0) - a(u_0)^2}{\xi^2 (\xi - a(u_0))^2} m(d\xi) \\
&\leq 2 \int_{[c, \infty)} \frac{a(u_0)}{\xi (c - a(u_0))^2} m(d\xi) = O(\sqrt{\alpha}).
\end{aligned} \tag{3.2.6}$$

Consequently

$$\frac{\sqrt{\alpha}}{a(u_0)} - \sigma_G = \frac{\frac{\alpha}{a(u_0)^2} - \sigma_G^2}{\frac{\sqrt{\alpha}}{a(u_0)} + \sigma_G} = O(\sqrt{\alpha}).$$

Now we put all the above together to prove the lemma.

$$\begin{aligned}
&\Psi_G(\alpha, 1) - \mu_G - 2\sigma_G\sqrt{\alpha} \\
&= \frac{\alpha}{a(u_0)} + \int_{[c, \infty)} \frac{1}{\xi - a(u_0)} m(d\xi) - \mu_G - 2\sigma_G\sqrt{\alpha} \\
&= \frac{\alpha}{a(u_0)} + \int_{[c, \infty)} \left[\frac{1}{\xi} + \frac{1}{\xi^2} a(u_0) + O(a(u_0)^2) \right] m(d\xi) - \mu_G - 2\sigma_G\sqrt{\alpha} \\
&= \sqrt{\alpha} \left(\frac{\sqrt{\alpha}}{a(u_0)} - \sigma_G \right) + \sigma_G a(u_0) \left(\sigma_G - \frac{\sqrt{\alpha}}{a(u_0)} \right) + \alpha \cdot O\left(\frac{a(u_0)^2}{\alpha}\right) \\
&= O(\alpha) \quad \text{as } \alpha \downarrow 0. \quad \square
\end{aligned}$$

We may also approach this problem from a different point of view. It uses the following theorem proved in [21]:

Theorem 3.2.2. *Two $/M/1$ queues in series are interchangeable: for any common input process $A(t)$, the output processes of $/M_1/1 \rightarrow /M_2/1$ and $/M_2/1 \rightarrow /M_1/1$ have the same distribution.*

In terms of the language in our last-passage models, the above theorem simply implies

that if we pick a realization of the exponential rates $\{\xi_j\}$, the distribution of $T(m, n)$ will not be affected if we exchange ξ_{j_1} and ξ_{j_2} for $0 \leq j_1 < j_2 \leq n$. We can start from this idea and reprove Theorem 3.2.1.

Proof. We first look at the case where the distribution m of the exponential rates has a finite state space $\{a_1, a_2, \dots, a_K\}$. Assume $m(\{a_k\}) = x_k, k = 1, 2, \dots, K$ with $\sum_k x_k = 1$. We now try to approximate $\Psi_G(\alpha, 1)$ by calculating $T(\lfloor N\alpha \rfloor, N)$ for very large N .

Fix an arbitrarily small $\delta > 0$. It is standard result that as N grows, the number of rows that have exponential rates a_k is in $[\lfloor N(1 - \delta)x_k \rfloor, \lfloor N(1 + \delta)x_k \rfloor]$ with probability converging to 1 exponentially fast.

Now we temporarily assume there are $\lfloor N(1 + \delta)x_k \rfloor$ rows with rate a_k for each k . Next we calculate the last-passage time from origin to $(\lfloor N\alpha \rfloor, \sum_k \lfloor N(1 + \delta)x_k \rfloor - 1)$. Because exponential variables are positive, this result will be no smaller than the actual $T(\lfloor N\alpha \rfloor, N)$ if N is large enough.

From Theorem 3.2.2 we can rearrange the rows without changing the distribution of the last-passage time. So without loss of generality we let the first $\lfloor N(1 + \delta)x_1 \rfloor$ rows have rates a_1 , the next $\lfloor N(1 + \delta)x_2 \rfloor$ rows have rates a_2 , and so on.

Now we have divided the first quadrant into K horizontal strips, each of which has underlying distribution $\exp(a_k)$. We select the optimal path π^* from the origin to $(\lfloor N\alpha \rfloor, \sum_k \lfloor N(1 + \delta)x_k \rfloor - 1)$, and record the points at which it exits each strip: $(M_1 - 1, \lfloor N(1 + \delta)x_1 \rfloor - 1), (M_1 + M_2 - 1, \lfloor N(1 + \delta)x_1 \rfloor + \lfloor N(1 + \delta)x_2 \rfloor - 1), \dots, (\lfloor N\alpha \rfloor, \sum_k \lfloor N(1 + \delta)x_k \rfloor - 1)$. Note $M_1 + M_2 + \dots + M_K = \lfloor N\alpha \rfloor + 1$.

Now we fix a small $\varepsilon > 0$, and define $r_k = \lceil \frac{M_k}{N\varepsilon} \rceil \varepsilon$. Then $\frac{M_k}{N} \leq r_k$, and $\sum_k r_k \leq \frac{1}{N}(\lfloor N\alpha \rfloor + 1) + K\varepsilon$.

We look at the intersection of π and the k -th strip. The weight of this portion of π is no more than the last-passage time from the origin to $(\lfloor Nr_k \rfloor, \lfloor N(1 + \delta)x_k \rfloor)$, with weights i.i.d. from $\exp(a_k)$. Now Theorem 1.6 from [10] can be applied and it shows that the maximal weight in this portion is bounded from above by $N[\frac{1}{a_k}(\sqrt{(1 + \delta)x_k} + \sqrt{r_k})^2 + \delta]$ with probability exponentially close to 1 in N . In fact for each k the rate function

$$\lim_{N \rightarrow \infty} \frac{1}{N} \log P\left(\left|\frac{1}{N} T_k(\lfloor Nr_k \rfloor, \lfloor N(1 + \delta)x_k \rfloor) - \frac{1}{a_k}(\sqrt{(1 + \delta)x_k} + \sqrt{r_k})^2\right| > \delta\right)$$

depends on the random variable r_k . However, $\{r_1, \dots, r_K\}$ has a finite state space $\{\varepsilon, 2\varepsilon, \dots, \lceil \frac{\alpha}{\varepsilon} \rceil \varepsilon\}$, so we still have a deterministic upper bound for the rate function.

So now we connect all strips and see that with probability converging to 1 exponentially fast, $T(\lfloor N\alpha \rfloor, N)$ is bounded from above by $N \sum_k [\frac{1}{a_k}(\sqrt{(1 + \delta)x_k} + \sqrt{r_k})^2 + \delta]$. By Cauchy-Schwarz inequality,

$$\begin{aligned} & \sum_k \left[\frac{1}{a_k} (\sqrt{(1 + \delta)x_k} + \sqrt{r_k})^2 \right] \\ &= \sum_k \frac{1}{a_k} (1 + \delta)x_k + 2 \sum_k \frac{1}{a_k} \sqrt{(1 + \delta)x_k r_k} + \sum_k \frac{r_k}{a_k} \\ &\leq \sum_k \frac{1}{a_k} (1 + \delta)x_k + 2 \sqrt{\sum_k \frac{1}{a_k^2} (1 + \delta)x_k \cdot \sum_k r_k} + \sum_k \frac{r_k}{c} \\ &\xrightarrow{N \rightarrow \infty} \sum_k \frac{1}{a_k} (1 + \delta)x_k + 2 \sqrt{\sum_k \frac{1}{a_k^2} (1 + \delta)x_k \sqrt{\alpha + K\varepsilon}} + \frac{1}{c} (\alpha + K\varepsilon). \end{aligned} \tag{3.2.7}$$

This shows that $\Psi_G(\alpha, 1) = \lim_{N \rightarrow \infty} T(\lfloor N\alpha \rfloor, N)/N$ is $\mathbf{P} - a.s.$ bounded from above by

$$\sum_k \frac{1}{a_k} x_k + 2 \sqrt{\sum_k \frac{1}{a_k^2} x_k \sqrt{\alpha}} + \frac{1}{c} \alpha$$

since we can make δ and ε arbitrarily small.

Then we can run a similar argument to find the lower bound: first assume there are $\lfloor N(1 - \delta)x_k \rfloor$ rows with rates a_k for each k and rearrange them to get K strips. We then pick a path $\pi \in \Pi(\lfloor N\alpha \rfloor, \sum_k \lfloor N(1 - \delta)x_k \rfloor - 1)$ in which we let

$$M_k = \lfloor N\alpha \cdot \frac{(1 - \delta)x_k a_k^{-2}}{\sum_k (1 - \delta)x_k a_k^{-2}} \rfloor$$

be the number of horizontal movements π makes in the k -th strip, and in each strip π chooses the path that gives the maximal weight.

With probability converging to 1 exponentially fast in N , the weight of π and hence $T(\lfloor N\alpha \rfloor, \sum_k \lfloor N(1 - \delta)x_k \rfloor - 1)$ is at least $N \sum_k \left[\frac{1}{a_k} (\sqrt{(1 - \delta)x_k} + \sqrt{M_k/N})^2 - \delta \right]$.

From our choice of M_k , we get

$$\begin{aligned} & \sum_k \frac{1}{a_k} (\sqrt{(1 - \delta)x_k} + \sqrt{M_k/N})^2 \\ & \geq \sum_k \frac{1}{a_k} (1 - \delta)x_k + 2 \sum_k \frac{1}{a_k} \sqrt{(1 - \delta)x_k M_k/N} \\ & \xrightarrow{N \rightarrow \infty} \sum_k \frac{1}{a_k} (1 - \delta)x_k + 2 \sqrt{\sum_k \frac{1}{a_k^2} (1 - \delta)x_k} \sqrt{\alpha}. \end{aligned}$$

Then we can use Borel-Cantelli Lemma, let δ go to 0, and claim the last line above is a lower bound of $\Psi_G(\alpha, 1)$.

It's worth noting that $\sum_k \frac{1}{a_k} x_k$ is actually the same as μ_G , the average of the means of the exponential distributions, and $\sum_k \frac{1}{a_k^2} x_k$ should be understood as σ_G^2 , the average

of the variances. Therefore we have already shown that

$$\mu_G + 2\sigma_G\sqrt{\alpha} \leq \Psi_G(\alpha, 1) \leq \mu_G + 2\sigma_G\sqrt{\alpha} + \frac{1}{c}\alpha \quad (3.2.8)$$

when the distribution m of exponential rates are supported on a finite space.

We can use discrete distributions to approximate any general distribution m . Let $\{\xi_j, X(i, j)\}$ be a realization of the exponential rates and the weights assigned to all lattice points. Choose an arbitrarily small $\varepsilon > 0$, couple them with $\{\rho_j, Y(i, j)\}$ in such a way that

$$\rho_j = \frac{1}{\lfloor \frac{1}{\xi_j \varepsilon} \rfloor \varepsilon}$$

and

$$Y(i, j) = \frac{\xi_j}{\rho_j} X(i, j).$$

Then $Y(i, j) \sim \exp(\rho_j)$ and ρ_0 is a random variable with a finite state space. In addition, we guarantee $Y(i, j) \leq X(i, j)$ and $0 \leq \frac{1}{\xi_j} - \frac{1}{\rho_j} < \varepsilon$.

Then we see

$$\begin{aligned} \Psi_G(\alpha, 1) &= \lim_{n \rightarrow \infty} \frac{1}{n} \mathbf{E} \max_{\pi \in \Pi(\lfloor n\alpha \rfloor, n)} \sum_{z \in \pi} X(z) \\ &\geq \lim_{n \rightarrow \infty} \frac{1}{n} \mathbf{E} \max_{\pi \in \Pi(\lfloor n\alpha \rfloor, n)} \sum_{z \in \pi} Y(z) \\ &\geq \mathbb{E} \frac{1}{\rho_0} + 2\sqrt{\mathbb{E} \frac{1}{\rho_0^2}} \sqrt{\alpha}. \end{aligned}$$

Letting ε go to 0 leads to $\Psi_G(\alpha, 1) \geq \mu_G + 2\sigma_G\sqrt{\alpha}$ by continuity.

For the other direction, we can define $\rho_j = \frac{1}{(\lfloor \frac{1}{\xi_j \varepsilon} \rfloor) \varepsilon}$ and repeat the same argument.

This gives the upper bound

$$\Psi_G(\alpha, 1) \leq \mu_G + 2\sigma_G\sqrt{\alpha} + \frac{1}{c}\alpha$$

and we have proved the theorem. \square

3.2.2 Estimate of $\Psi_G(1, \alpha)$

Next, we switch the two coordinates and estimate $\Psi_G(1, \alpha)$. Here we see how the tail of the random mean μ_0 creates different orders of magnitude for the α -dependent correction term. Qualitative properties of the limit shape depend on the tail of the distribution m at $c+$, and transitions occur where the integrals $\int_{[c, \infty)} (\xi - c)^{-2} m(d\xi)$ and $\int_{[c, \infty)} (\xi - c)^{-1} m(d\xi)$ blow up. ([20] also addressed this phenomenon.) These same regimes appear in our results below. For the case $\int_{[c, \infty)} (\xi - c)^{-2} m(d\xi) = \infty$ we make a precise assumption about the tail of the distribution of the random rate:

$$\exists \nu \in [-1, 1], \kappa > 0 \text{ such that } \lim_{\xi \searrow c} \frac{m[c, \xi]}{(\xi - c)^{\nu+1}} = \kappa. \quad (3.2.9)$$

The value $\nu = -1$ means that the bottom rate c has probability $m\{c\} = \kappa > 0$. Values $\nu < -1$ are of course not possible.

Theorem 3.2.3. *For the model with exponential distributions with i.i.d. random rates the limit Ψ_G has these asymptotics close to the x -axis.*

Case 1: $\int_{[c, \infty)} (\xi - c)^{-2} m(d\xi) < \infty$. Then there exists $\alpha_0 > 0$ such that

$$\Psi_G(1, \alpha) = c^{-1} + \alpha \int_{[c, \infty)} \frac{1}{\xi - c} m(d\xi) \quad \text{for } \alpha \in [0, \alpha_0]. \quad (3.2.10)$$

Case 2: (3.2.9) holds so that, in particular $\int_{[c,\infty)} (\xi - c)^{-2} m(d\xi) = \infty$. Then as $\alpha \searrow 0$,

$$\text{if } \nu \in (0, 1] \text{ then } \Psi_G(1, \alpha) = c^{-1} + \alpha \int_{[c,\infty)} \frac{1}{\xi - c} m(d\xi) + o(\alpha); \quad (3.2.11)$$

$$\text{if } \nu = 0 \text{ then } \Psi_G(1, \alpha) = c^{-1} - \kappa \alpha \log \alpha + o(\alpha \log \alpha); \quad (3.2.12)$$

$$\text{if } \nu \in [-1, 0) \text{ then } \Psi_G(1, \alpha) = c^{-1} + B \alpha^{\frac{1}{1-\nu}} + o(\alpha^{\frac{1}{1-\nu}}). \quad (3.2.13)$$

In statement (3.2.13) above $B = B(c, \kappa, \nu)$ is a constant whose explicit definition is in equation (3.2.20) in the proof below. The extreme case $\nu = -1$ is the one that matches up with Theorem 4.2.3.

Proof. Equation (3.2.2) gives

$$\Psi_G(1, \alpha) = \inf\{t \geq 0 : tg(\alpha/t) \geq 1\} = t(\alpha) = t. \quad (3.2.14)$$

That the infimum is achieved can be seen from (3.2.1).

Under Case 1 the critical value $u^* = \int_{[c,\infty)} c(\xi - c)^{-1} m(d\xi) < \infty$, and also

$$a'(u^* -) = \left\{ \int_{[c,\infty)} \frac{\xi}{(\xi - c)^2} m(d\xi) \right\}^{-1} > 0.$$

By the concavity of a and (3.2.1), for $0 \leq y \leq a'(u^* -)$ we have $g(y) = -yu^* + c$.

Consequently for small enough α

$$1 = tg(\alpha/t) = -\alpha c \int_{[c,\infty)} \frac{1}{\xi - c} m(d\xi) + ct$$

and equation (3.2.10) follows.

In Case 2 $a'(0+) > a'(u^*-) = 0$ and hence for small enough $\alpha > 0$ there exists a unique $u_0 \in (0, u^*)$ such that $a'(u_0) = \alpha/t$. Set $a_0 = a(u_0) \in (0, c)$. As $\alpha \searrow 0$, both $u_0 \nearrow u^*$ and $a_0 \nearrow c$. We have the equations

$$\begin{aligned} a'(u_0)^{-1} &= \int_{[c, \infty)} \frac{\xi}{(\xi - a_0)^2} m(d\xi) = \frac{t}{\alpha}, \quad 1 = tg(\alpha/t) = -\alpha u_0 + t a_0, \\ t &= \frac{1}{a} + \frac{\alpha u_0}{a_0} = \frac{1}{a_0} + \alpha \int_{[c, \infty)} \frac{1}{\xi - a_0} m(d\xi) \end{aligned} \quad (3.2.15)$$

and

$$\frac{1}{a_0^2} = \alpha \int_{[c, \infty)} \frac{1}{(\xi - a_0)^2} m(d\xi). \quad (3.2.16)$$

Assuming (3.2.9), start with $\nu \in (-1, 0) \cup (0, 1)$. For a small enough $\varepsilon > 0$ there are constants $0 < \kappa_1 < \kappa_2$ such that

$$\kappa_1(\xi - c)^{\nu+1} \leq m[c, \xi] \leq \kappa_2(\xi - c)^{\nu+1} \quad \text{for } \xi \in [c, c + \varepsilon] \quad (3.2.17)$$

and as $\varepsilon \searrow 0$ we can take $\kappa_1, \kappa_2 \rightarrow \kappa$. First we estimate $c - a_0$. Fix $\varepsilon > 0$.

$$\begin{aligned} \frac{1}{\alpha} &= a_0^2 \int_{[c, \infty)} \frac{1}{(\xi - a_0)^2} m(d\xi) = 2a_0^2 \int_c^\infty \frac{m[c, \xi]}{(\xi - a_0)^3} d\xi \\ &= 2a_0^2 \int_c^{c+\varepsilon} \frac{m[c, \xi]}{(\xi - a_0)^3} d\xi + C_1(\varepsilon) \end{aligned}$$

for a quantity $C_1(\varepsilon) = O(\varepsilon^{-2})$. The first term above can be bounded above and below

by (3.2.17), and we develop both bounds together for κ_i , $i = 1, 2$, as

$$\begin{aligned}
& 2\kappa_i a_0^2 \int_c^{c+\varepsilon} \frac{(\xi - c)^{\nu+1}}{(\xi - a_0)^3} d\xi + C_1(\varepsilon) \\
&= 2\kappa_i a_0^2 \int_c^{c+\varepsilon} \frac{[(\xi - a_0) - (c - a_0)]^{\nu+1}}{(\xi - a_0)^3} d\xi + C_1(\varepsilon) \\
&= 2\kappa_i a_0^2 \sum_{k=0}^{\infty} \binom{\nu+1}{k} (-1)^k (c - a_0)^k \int_c^{c+\varepsilon} (\xi - a_0)^{\nu-k-2} d\xi + C_1(\varepsilon) \\
&= 2\kappa_i a_0^2 \sum_{k=0}^{\infty} \binom{\nu+1}{k} (-1)^k (c - a_0)^k \frac{(c - a_0)^{\nu-k-1} - (c + \varepsilon - a_0)^{\nu-k-1}}{k - \nu + 1} + C_1(\varepsilon) \\
&= 2\kappa_i a_0^2 A_\nu (c - a_0)^{\nu-1} - 2\kappa_i a_0^2 \sum_{k=0}^{\infty} \binom{\nu+1}{k} \frac{(-1)^k}{k - \nu + 1} (c - a_0)^k (c + \varepsilon - a_0)^{\nu-k-1} + C_1(\varepsilon) \\
&= 2\kappa_i a_0^2 A_\nu (c - a_0)^{\nu-1} + C_1(\varepsilon).
\end{aligned} \tag{3.2.18}$$

$C_1(\varepsilon)$ changed of course in the last equality. In the next to last equality above we defined

$$A_\nu = \sum_{k=0}^{\infty} \binom{\nu+1}{k} \frac{(-1)^k}{k - \nu + 1}.$$

Rewrite the above development in the form

$$(c - a_0)^{1-\nu} = 2\kappa c^2 A_\nu \alpha + \alpha [2A_\nu (\kappa_i a_0^2 - \kappa c^2) + C_1(\varepsilon)(c - a_0)^{1-\nu}].$$

Now choose $\varepsilon = \varepsilon(\alpha) \searrow 0$ as $\alpha \searrow 0$ but slowly enough so that $C_1(\varepsilon)(c - a_0)^{1-\nu} \rightarrow 0$ as $\alpha \searrow 0$. Then also $\kappa_i a_0^2 \rightarrow \kappa c^2$ and we can write

$$c - a_0 = B_0 \alpha^{\frac{1}{1-\nu}} + o(\alpha^{\frac{1}{1-\nu}}) \tag{3.2.19}$$

with a new constant $B_0 = (2\kappa c^2 A_\nu)^{\frac{1}{1-\nu}}$.

Now consider the case $\nu \in (0, 1)$ which also guarantees $\int_{[c, \infty)} (\xi - c)^{-1} m(d\xi) < \infty$.

From (3.2.15) and (3.2.19) as $\alpha \searrow 0$

$$\begin{aligned} \Psi_G(1, \alpha) &= \frac{1}{a_0} + \alpha \int_{[c, \infty)} \frac{1}{\xi - a_0} m(d\xi) \\ &= \frac{1}{c} + \alpha \int_{[c, \infty)} \frac{1}{\xi - c} m(d\xi) + O(\alpha^{\frac{1}{1-\nu}}) + \alpha \left(\int_{[c, \infty)} \frac{1}{\xi - a_0} m(d\xi) - \int_{[c, \infty)} \frac{1}{\xi - c} m(d\xi) \right) \\ &= \frac{1}{c} + \alpha \int_{[c, \infty)} \frac{1}{\xi - c} m(d\xi) + o(\alpha). \end{aligned}$$

Next the case $\nu \in (-1, 0)$. The steps are similar to those above so we can afford to be sketchy.

$$\begin{aligned} \Psi_G(1, \alpha) &= \frac{1}{a_0} + \alpha \int_{[c, \infty)} \frac{1}{\xi - a_0} m(d\xi) \\ &= \frac{1}{c} + \frac{c - a_0}{c^2} + \frac{(c - a_0)^2}{c^2 a_0} + \alpha \int_c^{c+\varepsilon} \frac{m[c, \xi]}{(\xi - a_0)^2} d\xi + \alpha C_1(\varepsilon). \end{aligned}$$

Again, using (3.2.17) and proceeding as in (3.2.18), we develop an upper and a lower bound for the quantity above with distinct constants κ_i , $i = 1, 2$. After bounding $m[c, \xi]$ above and below with $\kappa_i(\xi - c)^{\nu+1}$ in the integral, write $(\xi - c)^{\nu+1} = ((\xi - a_0) - (c - a_0))^{\nu+1}$ and expand in power series.

$$\begin{aligned} &\frac{1}{c} + B_0 c^{-2} \alpha^{\frac{1}{1-\nu}} + o(\alpha^{\frac{1}{1-\nu}}) + \alpha \kappa_i \int_c^{c+\varepsilon} \frac{(\xi - c)^{\nu+1}}{(\xi - a_0)^2} d\xi + \alpha C_1(\varepsilon) \\ &= \frac{1}{c} + B_0 c^{-2} \alpha^{\frac{1}{1-\nu}} + o(\alpha^{\frac{1}{1-\nu}}) + \alpha \kappa_i (c - a_0)^\nu \sum_{k=0}^{\infty} \binom{\nu+1}{k} \frac{(-1)^k}{k - \nu} \\ &\quad + \alpha \kappa_i (c - a_0 + \varepsilon)^\nu \sum_{k=0}^{\infty} \binom{\nu+1}{k} \frac{(-1)^k}{\nu - k} \left(\frac{c - a_0}{c - a_0 + \varepsilon} \right)^k + \alpha C_1(\varepsilon) \\ &= \frac{1}{c} + B \alpha^{\frac{1}{1-\nu}} + o(\alpha^{\frac{1}{1-\nu}}) + A_{\nu,2} \alpha (\kappa_i - \kappa) (c - a_0)^\nu + \alpha C_1(\varepsilon). \end{aligned}$$

In the last equality the next to last term with the $\sum_{k=0}^{\infty}$ sum was subsumed in the $\alpha C_1(\varepsilon)$ term. Then we introduced new constants

$$A_{\nu,2} = \sum_{k=0}^{\infty} \binom{\nu+1}{k} \frac{(-1)^k}{k-\nu} \quad \text{and} \quad B = B_0 c^{-2} + \kappa B_0^{\nu} A_{\nu,2}. \quad (3.2.20)$$

As before, by letting $\varepsilon = \varepsilon(\alpha) \searrow 0$ slowly enough as $\alpha \searrow 0$ we can extract $\Psi_G(1, \alpha) = c^{-1} + B\alpha^{\frac{1}{1-\nu}} + o(\alpha^{\frac{1}{1-\nu}})$ from the above bounds.

It remains to treat the cases $\nu = -1, 0, 1$ where integration of the type done in (3.2.18) is elementary. We omit the details. \square

Chapter 4

Universality results

4.1 Limiting shape near the y -axis

Now we turn to the results on the form of the limit shape at the boundary for a general process $\{F_j\}_{j \in \mathbb{Z}_+}$. As explained in the introduction, for $\Psi(\alpha, 1)$ we find a universal form as $\alpha \searrow 0$. In addition to the earlier assumptions (2.2.2) and (2.2.3), we need similar control of the left tail of the distributions:

$$\int_{-\infty}^0 (\mathbb{E}[F_0(x)])^{1/2} dx < \infty \quad (4.1.1)$$

and

$$\int_{-\infty}^0 \operatorname{ess\,sup}_{\mathbb{P}} F_0(x) dx < \infty. \quad (4.1.2)$$

Let us point out that (2.2.2) and (4.1.1) together guarantee $\mathbb{E}|X(z)|^2 < \infty$. Let $\mu_j = \mu(F_j)$ and $\sigma_j^2 = \sigma^2(F_j)$ denote the mean and variance of distribution F_j . These are random variables under \mathbb{P} with expectations $\mu = \mathbb{E}(\mu_0)$ and $\sigma^2 = \mathbb{E}(\sigma_0^2)$. Here is our main theorem of this chapter:

Theorem 4.1.1. *Assume the process $\{F_j\}$ is i.i.d. under \mathbb{P} , and satisfies tail assumptions (2.2.2), (2.2.3), (4.1.1) and (4.1.2). Then, as $\alpha \downarrow 0$, $\Psi(\alpha, 1) = \mu + 2\sigma\sqrt{\alpha} + o(\sqrt{\alpha})$.*

4.1.1 Proof of Theorem 4.1.1

We start the proof with the first lemma, which enables us to compare the last-passage time limits with different underlying distribution sequences. let $\{F_j\}$ and $\{G_j\}$ be ergodic sequences of distributions defined on a common probability space under probability measure \mathbb{P} . In a later step of the proof we need to assume $\{F_j\}$ i.i.d. Assume that both processes $\{F_j\}$ and $\{G_j\}$ satisfy the assumptions made in Theorem 4.1.1. With some abuse of notation we label the time constants, means, and even random weights associated to the processes $\{F_j\}$ and $\{G_j\}$ with subscripts F and G . So for example $\mu_F = \mathbb{E}(\int x dF_0(x))$. The symbolic subscripts F and G should not be confused with the random distributions F_j and G_j assigned to the rows of the lattice. We write $\Psi_{Ber([G(x)-F(x)]_+)}$ for the limit of a Bernoulli model with weight distributions $P(X(i, j) = 1) = (G_j(x) - F_j(x))_+ = 1 - P(X(i, j) = 0)$ where x is a fixed parameter. An analogous convention will be used for other Bernoulli models along the way.

Lemma 4. *Assume $\{F_j\}$ and $\{G_j\}$ satisfy (2.2.2), (2.2.3), (4.1.1) and (4.1.2). Then for $\alpha > 0$,*

$$\begin{aligned} & |\Psi_F(\alpha, 1) - \Psi_G(\alpha, 1) - (\mu_F - \mu_G)| \\ & \leq 8\sqrt{\alpha} \int_{-\infty}^{+\infty} \left(\mathbb{E}|G_0(x) - F_0(x)| \right)^{1/2} dx + \alpha \int_{-\infty}^{+\infty} \text{ess sup}_{\mathbb{P}} |F_0(x) - G_0(x)| dx. \end{aligned} \tag{4.1.3}$$

Proof. The right-hand side of (4.1.3) is finite under the assumptions on $\{F_j\}$ and $\{G_j\}$. Couple the F_j and G_j distributed weights in a standard way. Let $\{u(z) : z = (i, j) \in \mathbb{Z}_+^2\}$ be i.i.d. Uniform(0, 1) random variables. Set $X_F(z) = F_j^{-1}(u(z))$, where $F_j^{-1}(u) = \sup\{x : F_j(x) < u\}$, and similarly $X_G(z) = G_j^{-1}(u(z))$. Write \mathbf{E} for expectation over the

entire probability space of distributions and weights.

The following equality will be useful when we relate arbitrary random variables to Bernoulli variables:

$$X_F(z) - X_G(z) = \int_{-\infty}^{+\infty} \left\{ I(X_G(z) \leq x < X_F(z)) - I(X_F(z) \leq x < X_G(z)) \right\} dx.$$

Now we compare $\Psi_F(\alpha, 1)$ and $\Psi_G(\alpha, 1)$:

$$\begin{aligned} & \Psi_F(\alpha, 1) - \Psi_G(\alpha, 1) \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \mathbf{E} \max_{\pi \in \Pi(\lfloor \alpha n \rfloor, n)} \sum_{z \in \pi} X_F(z) - \lim_{n \rightarrow \infty} \frac{1}{n} \mathbf{E} \max_{\pi \in \Pi(\lfloor \alpha n \rfloor, n)} \sum_{z \in \pi} X_G(z) \\ &\leq \overline{\lim}_{n \rightarrow \infty} \frac{1}{n} \mathbf{E} \max_{\pi \in \Pi(\lfloor \alpha n \rfloor, n)} \sum_{z \in \pi} (X_F(z) - X_G(z)) \\ &= \overline{\lim}_{n \rightarrow \infty} \frac{1}{n} \mathbf{E} \max_{\pi \in \Pi(\lfloor \alpha n \rfloor, n)} \sum_{z \in \pi} \int_{-\infty}^{+\infty} \left\{ I(X_G(z) \leq x < X_F(z)) - I(X_F(z) \leq x < X_G(z)) \right\} dx \\ &\leq \overline{\lim}_{n \rightarrow \infty} \frac{1}{n} \mathbf{E} \int_{-\infty}^{+\infty} \max_{\pi \in \Pi(\lfloor \alpha n \rfloor, n)} \sum_{z \in \pi} \left\{ I(X_G(z) \leq x < X_F(z)) - I(X_F(z) \leq x < X_G(z)) \right\} dx. \end{aligned}$$

We check that Fubini allows us to interchange the integral and the expectation. Since F and G are interchangeable it is enough to consider the first indicator function from above. Let a be an integer $\geq \alpha$.

$$\begin{aligned} & \int_{-\infty}^{+\infty} \frac{1}{n} \mathbf{E} \max_{\pi \in \Pi(\lfloor \alpha n \rfloor, n)} \sum_{z \in \pi} I(X_G(z) \leq x < X_F(z)) dx \\ &\leq \int_{-\infty}^{+\infty} \sup_n \frac{1}{n} \mathbf{E} \max_{\pi \in \Pi(an, n)} \sum_{z \in \pi} I(X_G(z) \leq x < X_F(z)) dx = \int_{-\infty}^{+\infty} \Psi_{Ber([G(x)-F(x)]_+)}(a, 1) dx \\ &\leq \int_{-\infty}^{+\infty} \left(\mathbb{E}|G_0(x) - F_0(x)| + 4\sqrt{a}(\mathbb{E}|G_0(x) - F_0(x)|)^{1/2} + a \operatorname{ess\,sup}_{\mathbb{P}} |G_0(x) - F_0(x)| \right) dx \\ &< \infty \end{aligned}$$

by estimate (3.1.7) and the finiteness of the right-hand side of (4.1.3). Continue from the limit above by applying Fubini. Then take the limit inside the dx -integral by dominated convergence, justified by the n -uniformity in the bound above. Finally apply again the Bernoulli estimate (3.1.7).

$$\begin{aligned}
& \Psi_F(\alpha, 1) - \Psi_G(\alpha, 1) \\
& \leq \overline{\lim}_{n \rightarrow \infty} \int_{-\infty}^{+\infty} \frac{1}{n} \mathbf{E} \max_{\pi \in \Pi(\lfloor \alpha n \rfloor, n)} \sum_{z \in \pi} \left\{ I(X_G(z) \leq x < X_F(z)) - I(X_F(z) \leq x < X_G(z)) \right\} dx \\
& \leq \int_{-\infty}^{+\infty} \lim_{n \rightarrow \infty} \frac{1}{n} \left\{ \mathbf{E} \max_{\pi \in \Pi(\lfloor \alpha n \rfloor, n)} \sum_{z \in \pi} I(X_G(z) \leq x < X_F(z)) \right. \\
& \quad \left. + \mathbf{E} \max_{\pi \in \Pi(\lfloor \alpha n \rfloor, n)} \sum_{z \in \pi} (1 - I(X_F(z) \leq x < X_G(z))) - \sum_{z \in \pi} 1 \right\} dx \\
& = \int_{-\infty}^{+\infty} \left\{ \Psi_{Ber([G(x) - F(x)]_+)}(\alpha, 1) + \Psi_{Ber(1 - [F(x) - G(x)]_+)}(\alpha, 1) - (1 + \alpha) \right\} dx \\
& \leq \int_{-\infty}^{+\infty} \left\{ \mathbb{E}(G_0(x) - F_0(x))_+ + 1 - \mathbb{E}(F_0(x) - G_0(x))_+ \right. \\
& \quad \left. + 4\sqrt{\alpha} \left(\sqrt{\mathbb{E}(G_0(x) - F_0(x))_+} + \sqrt{\mathbb{E}(F_0(x) - G_0(x))_+} \right) \right. \\
& \quad \left. + \alpha \left(\operatorname{ess\,sup}_{\mathbb{P}}[G_0(x) - F_0(x)]_+ + 1 - \operatorname{ess\,inf}_{\mathbb{P}}[F_0(x) - G_0(x)]_+ \right) - (1 + \alpha) \right\} dx \\
& \leq (\mu_F - \mu_G) + 8\sqrt{\alpha} \int_{-\infty}^{+\infty} \sqrt{\mathbb{E}|F_0(x) - G_0(x)|} dx + \alpha \int_{-\infty}^{+\infty} \operatorname{ess\,sup}_{\mathbb{P}} |G_0(x) - F_0(x)| dx.
\end{aligned}$$

Interchanging F and G in the above inequality gives the bound from the other direction and concludes the proof. \square

For a while we make a convenient assumption that the weights are uniformly bounded, so for a constant $M < \infty$,

$$\mathbb{P}\{F_0(-M) = 0 \text{ and } F_0(M) = 1\} = 1, \quad (4.1.4)$$

then it's easy to see

$$\sigma^2(F_0) \leq M^2 \quad \mathbb{P}\text{-a.s} \quad (4.1.5)$$

and the conditions assumed for Theorem 4.1.1 are trivially satisfied by the uniform boundedness.

Henceforth $r = r(\alpha)$ denotes a positive integer-valued function such that $r(\alpha) \nearrow \infty$ as $\alpha \searrow 0$. Tile the lattice with $1 \times r$ blocks $B_r(x, y) = \{(x, ry + k) : k = 0, 1, \dots, r-1\}$ for $(x, y) \in \mathbb{Z}_+^2$. A coarse-grained last-passage model is defined by adding up the weights in each block:

$$X_r(z) = \sum_{v \in B_r(z)} X(v).$$

The distribution of the new weight $X_r(i, j)$ on row $j \in \mathbb{Z}_+$ of the rescaled lattice is the convolution $F_{r,j} = F_{rj} * F_{rj+1} * \dots * F_{rj+r-1}$.

We repeat Lemma 4.4 from [13] with a sketch of the argument.

Lemma 5. *Let $\Psi_F(x, y)$ and $\Psi_{F_r}(x, y)$ be the last-passage time functions obtained by using F_j and $F_{r,j}$ as the distributions on the j th row, respectively. If $r \rightarrow \infty$ and $r\sqrt{\alpha} \rightarrow 0$ as $\alpha \downarrow 0$, then*

$$\lim_{\alpha \downarrow 0} \frac{1}{\sqrt{\alpha}} \left| \Psi_F(\alpha, 1) - \frac{1}{r} \Psi_{F_r}(\alpha r, 1) \right| = 0.$$

Proof. Given a path $\pi \in \Pi(m, nr-1)$, consider all the blocks that it intersects; this gives a path $\tilde{\pi} \in \Pi(m, n-1)$ in the rescaled lattice. This path contains almost all the points in π , with the possible exception at the end point. For example, π may contain the point $(m, nr-2)$, but $\tilde{\pi}$ does not if $r \geq 2$. So there are at most $r(m+n-1) - (m+nr-1) = (m-1)(r-1)$ points in $\tilde{\pi}$ but not in π , and at most r points in π but not in $\tilde{\pi}$.

So $|(\cup_{z \in \tilde{\pi}} B_r(z)) \Delta \pi| \leq mr$ when r is large. Then by (4.1.4)

$$\left| \max_{\pi \in \Pi(m, nr-1)} \sum_{z \in \pi} X(z) - \max_{\tilde{\pi} \in \Pi(m, n-1)} \sum_{z \in \tilde{\pi}} X_r(z) \right| \leq mrM.$$

Let $m = \lfloor \alpha nr \rfloor$, divide through by nr , and take limits, finally we arrive at

$$\lim_{\alpha \downarrow 0} \frac{1}{\sqrt{\alpha}} |\Psi(\alpha, 1) - \frac{1}{r} \Psi(\alpha r, 1)| \leq \lim_{\alpha \downarrow 0} Mr\sqrt{\alpha} = 0 \quad (4.1.6)$$

□

Let $\mu_{r,y}$ and $V_{r,y}$ be the mean and variance of $F_{r,y}$:

$$\mu_{r,y} = \sum_{i=0}^{r-1} \mu_{ry+i}, \quad \text{and} \quad V_{r,y} = \sum_{i=0}^{r-1} \sigma_{ry+i}^2.$$

Let $\Phi_{r,y}$ be the distribution function of the normal $\mathcal{N}(\mu_{r,y}, V_{r,y})$ distribution, and $\tilde{\Phi}_{r,y}$ the distribution function of $\mathcal{N}(r\mu_F, V_{r,y})$. The difference between $\Phi_{r,y}$ and $\tilde{\Phi}_{r,y}$ is that the latter has a non-random mean. We shall also find it convenient to use $\{X_j\}$ as a sequence of independent variables with (random) distributions $X_j \sim F_j$. For the next lemma we need to assume $\{F_j\}$ an i.i.d. sequence under \mathbb{P} .

As in [13], a key step in the proof is the replacement of the rescaled weights with Gaussian weights, which is undertaken in the next lemma.

Lemma 6. *Assume $\{F_j\}$ i.i.d. under \mathbb{P} . If $r \rightarrow \infty$ and $r\sqrt{\alpha} \rightarrow 0$ as $\alpha \downarrow 0$, then*

$$\lim_{\alpha \downarrow 0} \frac{1}{r\sqrt{\alpha}} |\Psi_{F_r}(\alpha r, 1) - \Psi_{\Phi_r}(\alpha r, 1)| = 0. \quad (4.1.7)$$

Remark 7. *The following proof contains many inequalities where we need to use letters*

to denote proper constants. For simplicity all constants that does not depend on r, α and M are subsumed in a single notation C .

Technically we could also subsume M into C and the proof of Theorem 4.1.1 is not affected. However, we write M explicitly to help us prove the next theorem 4.1.2.

Proof. We will use (4.1.3) for the processes $\{F_{r,y}\}_{y \in \mathbb{Z}_+}$ and $\{\Phi_{r,y}\}_{y \in \mathbb{Z}_+}$ and with α replaced by αr . On the right-hand side there are two terms. We will handle the second term first.

To estimate the second integral, we discuss over the value of x . For $x \geq 2rM$, we note that $F_{r,y}(x) = 1$ since we assume (4.1.4). Now we turn to $\Phi_{r,y}(x)$. We quote Theorem 1.4 from [4] and get

$$\begin{aligned}
\Phi_{r,y}(x) &= 1 - \int_x^\infty \frac{1}{\sqrt{2\pi V_{r,y}}} \exp\left(-\frac{(s - \mu_{r,y})^2}{2V_{r,y}}\right) ds \\
&\geq 1 - \frac{\sqrt{V_{r,y}}}{\sqrt{2\pi}(x - \mu_{r,y})} \exp\left(-\frac{(x - \mu_{r,y})^2}{2V_{r,y}}\right) \\
&\geq 1 - \frac{\sqrt{rM^2}}{\sqrt{2\pi}(2rM - rM)} \exp\left(-\frac{(x - rM)^2}{2rM^2}\right) \\
&= 1 - \frac{1}{\sqrt{2\pi r}} \exp\left(-\frac{(x - rM)^2}{2rM^2}\right).
\end{aligned} \tag{4.1.8}$$

This gives

$$|F_{r,y}(x) - \Phi_{r,y}(x)| \leq \frac{1}{\sqrt{2\pi r}} \exp\left(-\frac{(x - rM)^2}{2rM^2}\right) \tag{4.1.9}$$

for $x > 2rM$. From symmetry a similar inequality

$$|F_{r,y}(x) - \Phi_{r,y}(x)| \leq \frac{1}{\sqrt{2\pi r}} \exp\left(-\frac{(x + rM)^2}{2rM^2}\right) \tag{4.1.10}$$

holds for $x < -2rM$.

Now we are ready to claim

$$\begin{aligned}
& \alpha r \int_{-\infty}^{+\infty} \operatorname{ess\,sup}_{\mathbb{P}} |F_{r,0}(x) - \Phi_{r,0}(x)| dx \\
& \leq \alpha r \left\{ \int_{-\infty}^{-2rM} \frac{1}{\sqrt{2\pi r}} \exp\left(-\frac{(x+rM)^2}{2rM^2}\right) dx + \int_{-2rM}^{2rM} 1 \cdot dx \right. \\
& \quad \left. + \int_{2rM}^{+\infty} \frac{1}{\sqrt{2\pi r}} \exp\left(-\frac{(x-rM)^2}{2rM^2}\right) dx \right\} \\
& \leq C\alpha r \left\{ M + 4rM \right\} \leq CM\alpha r^2.
\end{aligned}$$

Next we estimate the first term on the right hand side of (4.1.3). We will need Theorem 5.17 of [14], which states that if independent mean 0 random variables X_1, X_2, X_3, \dots all have finite third moments, then they satisfy the estimate

$$\left| P\left\{ B_r^{-1/2} \sum_{i=1}^r X_i \leq x \right\} - \Phi(x) \right| \leq A \frac{\sum_{i=1}^r E|X_i|^3}{B_r^{3/2}} (1 + |x|)^{-3}, \quad x \in \mathbb{R},$$

where $B_r = \sum_{i=1}^r \operatorname{Var}(X_i)$, Φ is the standard normal distribution function, and A is a constant that is independent of the distribution functions of X_1, X_2, \dots, X_r .

Recall that we assume $\{F_j\}$ i.i.d. under \mathbb{P} , so $\sigma^2(F_j)$ are i.i.d. random variables. For an arbitrary $0 < \varepsilon < 1$, say $\varepsilon = \frac{1}{2}$, we define U_r as the event $\sum_{i=0}^{r-1} \sigma_{ry+i}^2 \geq r(1-\varepsilon)\mathbb{E}\sigma_0^2 = \frac{1}{2}r\mathbb{E}\sigma_0^2$. Then it is standard result that $\mathbb{P}(U_r)$ converges to 1 exponentially fast as r goes to infinity.

With probability $\mathbb{P}(U_r)$ we get

$$\begin{aligned}
|F_{r,y}(x) - \Phi_{r,y}(x)| & \leq A \frac{\sum_{i=0}^{r-1} E|X_{ry+i} - \mu_{ry+i}|^3}{(\sum_{i=0}^{r-1} \sigma_{ry+i}^2)^{3/2}} (1 + V_{r,y}^{-1/2}|x - \mu_{r,y}|)^{-3} \\
& \leq \frac{CM^3}{\sqrt{r}} (1 + M^{-1}r^{-1/2}|x - \mu_{r,y}|)^{-3}
\end{aligned} \tag{4.1.11}$$

where the second inequality used the assumptions $P(|X_i| \leq M) = 1$ and the property of U_r , and C is a proper constant that depends on A and $\mathbb{E}\sigma_0^2$.

Next we note this trick using Cauchy-Schwarz inequality: for a probability density f on \mathbb{R} and a function $H \geq 0$,

$$\int \sqrt{H} dx = \int f^{1/2} \sqrt{f^{-1}H} dx \leq \left(\int f^{-1}H dx \right)^{1/2}.$$

Then we get

$$\begin{aligned} & \sqrt{\alpha r} \int_{-\infty}^{+\infty} \left(\mathbb{E}|F_{r,0}(x) - \Phi_{r,0}(x)| \right)^{1/2} dx \\ & \leq \sqrt{\alpha r} \left\{ \int_{-\infty}^{+\infty} \frac{1}{f(x)} \mathbb{E}|F_{r,0}(x) - \Phi_{r,0}(x)| dx \right\}^{1/2} \\ & = \sqrt{\alpha r} \left\{ \mathbb{E} \int_{-\infty}^{+\infty} \frac{1}{f(x)} |F_{r,0}(x) - \Phi_{r,0}(x)| dx \right\}^{1/2}. \end{aligned} \tag{4.1.12}$$

For the calculation below take $\delta > 0$ and $f(x) = c_1(1 + |x - r\mu_F|^{1+\delta})^{-1}$ for the right constant $c_1 = c_1(\delta)$ to make $\int_{-\infty}^{\infty} f(x)dx = 1$. Again factors that depend on δ are subsumed in a constant C in each of the following steps.

Over the event U_r ,

$$\begin{aligned} & \int_{-\infty}^{+\infty} \frac{1}{f(x)} |F_{r,0}(x) - \Phi_{r,0}(x)| dx \\ & \leq \frac{CM^3}{\sqrt{r}} \int_{-\infty}^{+\infty} (1 + |x - r\mu_F|^{1+\delta}) \left(1 + \frac{|x - \mu_{r,0}|}{M\sqrt{r}} \right)^{-3} dx \\ & \text{by a change of variables } x = \mu_{r,0} + yM\sqrt{r} \\ & = CM^4 \int_{-\infty}^{+\infty} \frac{1 + |\mu_{r,0} - r\mu_F + yM\sqrt{r}|^{1+\delta}}{(1 + |y|)^3} dy \\ & \leq CM^4 (|\mu_{r,0} - r\mu_F|^{1+\delta} + M^{1+\delta} r^{(1+\delta)/2}). \end{aligned} \tag{4.1.13}$$

Over the event U_r^c we use (4.1.9) and (4.1.10) to bound the integral

$$\begin{aligned}
& \int_{-\infty}^{+\infty} \frac{1}{f(x)} |F_{r,0}(x) - \Phi_{r,0}(x)| dx \\
& \leq \int_{-\infty}^{-2rM} (1 + |x - r\mu_F|^{1+\delta}) \frac{1}{\sqrt{2\pi}r} \exp\left(-\frac{(x + rM)^2}{2rM^2}\right) dx + \int_{-2rM}^{2rM} (1 + |x - r\mu_F|^{1+\delta}) dx \\
& + \int_{2rM}^{\infty} (1 + |x - r\mu_F|^{1+\delta}) \frac{1}{\sqrt{2\pi}r} \exp\left(-\frac{(x - rM)^2}{2rM^2}\right) dx.
\end{aligned}$$

We use a change of variables $y = \frac{x+rM}{\sqrt{r}M}$ and the first term above

$$\begin{aligned}
& = \int_{-\infty}^{-\sqrt{r}} (1 + |\sqrt{r}My - rM - r\mu_F|^{1+\delta}) \frac{M}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right) dy \\
& \leq C \int_{-\infty}^{-\sqrt{r}} (|\sqrt{r}My|^{1+\delta} + |rM + r\mu_F|^{1+\delta}) \frac{M}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right) dy \\
& \leq C \int_{-\infty}^{-\sqrt{r}} M^{1+\delta} r^{\frac{1+\delta}{2}} |y|^{1+\delta} \frac{M}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right) dy \\
& + C \int_{-\infty}^{-\sqrt{r}} r^{1+\delta} M^{1+\delta} \frac{M}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right) dy \\
& \leq C(r^{(1+\delta)/2} M^{2+\delta} + r^{1+\delta} M^{2+\delta}) \leq Cr^{1+\delta} M^{2+\delta}
\end{aligned}$$

The third term follow the same upper bounds.

The second term is simply bounded by $C \cdot rM \cdot (rM)^{1+\delta} = CM^{2+\delta}r^{2+\delta}$. Then

$$\int_{-\infty}^{+\infty} \frac{1}{f(x)} |F_{r,0}(x) - \Phi_{r,0}(x)| dx \leq CM^{2+\delta}r^{2+\delta}. \quad (4.1.14)$$

Continue from (4.1.12), and keep in mind that $\mathbb{P}(U_r^c)$ decays exponentially as r grows,

we have

$$\begin{aligned}
& \sqrt{\alpha r} \int_{-\infty}^{+\infty} \left(\mathbb{E} |F_{r,0}(x) - \Phi_{r,0}(x)| \right)^{1/2} dx \\
& \leq \sqrt{\alpha r} \left\{ \mathbb{E} \int_{-\infty}^{+\infty} \frac{1}{f(x)} |F_{r,0}(x) - \Phi_{r,0}(x)| dx \right\}^{1/2} \\
& \leq C \sqrt{\alpha r} \left\{ M^4 \mathbb{E} |\mu_{r,0} - r\mu_F|^{1+\delta} + M^{5+\delta} r^{(1+\delta)/2} + M^{2+\delta} r^{2+\delta} \mathbb{P}(U_r^c) \right\}^{1/2} \\
& \leq CM^{(5+\delta)/2} \alpha^{1/2} r^{(3+\delta)/4}.
\end{aligned} \tag{4.1.15}$$

Here we used the fact that $\mu_{r,0} - \mu_F$ is a sum of independent bounded mean-zero variables, so

$$\begin{aligned}
\mathbb{E} |\mu_{r,0} - r\mu_F|^{1+\delta} & \leq \left(\mathbb{E} |\mu_{r,0} - r\mu_F|^2 \right)^{(1+\delta)/2} \\
& = [r \mathbb{E} (\mu_0 - \mu_F)^2]^{(1+\delta)/2} = CM^{1+\delta} r^{(1+\delta)/2}.
\end{aligned}$$

To summarize, with these estimates and (4.1.3) we have

$$\begin{aligned}
& \frac{1}{r\sqrt{\alpha}} |\Psi_{F_r}(\alpha r, 1) - \Psi_{\Phi_r}(\alpha r, 1)| \\
& \leq \frac{C}{r\sqrt{\alpha}} (M^{(5+\delta)/2} \alpha^{1/2} r^{(3+\delta)/4} + M\alpha r^2) \\
& = C(M^{(5+\delta)/2} r^{(-1+\delta)/4} + Mr\sqrt{\alpha}).
\end{aligned} \tag{4.1.16}$$

By choosing $\delta < 1$, then assumptions $r \rightarrow \infty$ and $r\sqrt{\alpha} \rightarrow 0$ make this vanish as $\alpha \rightarrow 0$.

The proof is completed. \square

Remark 8. *The proof presented above distinguished the two events U_r and U_r^c because (4.1.11) only works when $V_{r,y} = \sum_{j=0}^{r-1} \sigma_j^2$ can be bounded away from zero. Therefore on U_r^c we used a different approach.*

We can actually find an alternative proof of this lemma. In addition to (4.1.4) let's also assume that variances are uniformly bounded away from zero, i.e. for a constant $0 < c_0 < \infty$,

$$\mathbb{P}\{\sigma^2(F_0) \geq c_0\} = 1. \quad (4.1.17)$$

Note that then $c_0 \leq \sigma^2(F_0) \leq M^2$. A direct consequence of this is that now (4.1.11) and (4.1.13) holds $\mathbb{P} - a.s.$, so

$$\sqrt{\alpha r} \int_{-\infty}^{+\infty} \left(\mathbb{E} |F_{r,0}(x) - \Phi_{r,0}(x)| \right)^{1/2} dx \leq C \alpha^{1/2} r^{(3+\delta)/4}.$$

Note that in equations above and below we subsume M into C for simplicity since we will not come back to them any more. For the second term on the right in (4.1.3),

$$\begin{aligned} \alpha r \int_{-\infty}^{+\infty} \operatorname{ess\,sup}_{\mathbb{P}} |F_{r,0}(x) - \Phi_{r,0}(x)| dx &\leq C \alpha r^{1/2} \int_{-\infty}^{+\infty} \operatorname{ess\,sup}_{\mathbb{P}} \left(1 + \frac{|x - \mu_{r,y}|}{M\sqrt{r}} \right)^{-3} dx \\ &\leq C \alpha r^{1/2} \left\{ \int_{-\infty}^{-rM} \left(1 + \frac{-rM - x}{M\sqrt{r}} \right)^{-3} dx + \int_{-rM}^{rM} dx + \int_{rM}^{+\infty} \left(1 + \frac{x - rM}{M\sqrt{r}} \right)^{-3} dx \right\} \\ &= C \alpha r^{1/2} \left\{ M\sqrt{r} \int_1^{\infty} u^{-3} du + 2rM + M\sqrt{r} \int_1^{\infty} u^{-3} du \right\} \\ &= C \alpha r^{1/2} (M\sqrt{r} + 2rM) \leq C \alpha r^{3/2}. \end{aligned}$$

Therefore we see (4.1.16) still holds and Lemma 6 is proved under (4.1.17). However, we eventually we have to show Theorem 4.1.1 without (4.1.17), so then we try to lift this assumption.

For $\varepsilon > 0$, let $\{W(z)\}$ be i.i.d weights with distribution H defined by $P(W(z) = \pm\varepsilon) = 1/2$. Let $\tilde{F}_j = F_j * H$ be the distribution of the weight $\tilde{X}(i, j) = X(i, j) + W(i, j)$. Let Ψ_H and $\Psi_{\tilde{F}}$ be the time constants of the last-passage models with weights $\{W(z)\}$ and

$\{\tilde{X}(z)\}$, respectively. The Bernoulli bound (3.1.7) gives the estimate $\Psi_H(x, y) \leq 4\varepsilon\sqrt{xy}$.

The corresponding last-passage times satisfy

$$T_{\tilde{F}}(z) - T_H(z) \leq T_F(z) \leq T_{\tilde{F}}(z) + \hat{T}_H(z)$$

where $\hat{T}_H(z)$ uses the weights $-W(z)$. In the limit

$$\Psi_{\tilde{F}}(\alpha, 1) - 4\varepsilon\sqrt{\alpha} \leq \Psi_F(\alpha, 1) \leq \Psi_{\tilde{F}}(\alpha, 1) + 4\varepsilon\sqrt{\alpha}. \quad (4.1.18)$$

Since $\sigma^2(\tilde{F}_j) = \sigma^2(F_j) + \varepsilon^2$, $\{\tilde{F}_j\}$ satisfies (4.1.17). Once $\{\tilde{F}_j\}_{j \in \mathbb{Z}_+}$ satisfies (4.1.25), then so does $\{F_j\}_{j \in \mathbb{Z}_+}$, because $\mu_{\tilde{F}} = \mu_F$ and $\varepsilon > 0$ can be arbitrarily small.

After the discussion of Lemma 6 we make a further approximation that puts us in the situation where all sites have normal variables with the same mean.

Lemma 9. *Let Ψ_{Φ_r} and $\Psi_{\tilde{\Phi}_r}$ be defined as before, and again $r\sqrt{\alpha} \rightarrow 0$ as $\alpha \rightarrow 0$. Then*

$$\lim_{\alpha \downarrow 0} \frac{1}{r\sqrt{\alpha}} |\Psi_{\Phi_r}(\alpha r, 1) - \Psi_{\tilde{\Phi}_r}(\alpha r, 1)| = 0.$$

Proof. For $z = (i, j) \in \mathbb{Z}_+^2$, let $X^{(r)}(z)$ have distribution $\Phi_{r,j}$ so that $\tilde{X}^{(r)}(z) = X^{(r)}(z) -$

$\mu_{r,j} + r\mu_F$ has distribution $\tilde{\Phi}_{r,j}$. Now estimate:

$$\begin{aligned}
\Psi_{\tilde{\Phi}_r}(\alpha r, 1) &= \lim_{n \rightarrow \infty} \frac{1}{n} \max_{\pi \in \Pi(\lfloor \alpha n r \rfloor, n)} \sum_{z \in \pi} \tilde{X}^{(r)}(z) \\
&\leq \lim_{n \rightarrow \infty} \frac{1}{n} \max_{\pi \in \Pi(\lfloor \alpha n r \rfloor, n)} \sum_{z \in \pi} X^{(r)}(z) + \lim_{n \rightarrow \infty} \frac{1}{n} \max_{\pi \in \Pi(\lfloor \alpha n r \rfloor, n)} \sum_{z \in \pi} (-\mu_{r,j} + r\mu_F) \\
&\leq \Psi_{\Phi_r}(\alpha r, 1) + \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^n (-\mu_{r,j} + r\mu_F) + \lim_{n \rightarrow \infty} \frac{1}{n} 2Mr \cdot \lfloor \alpha n r \rfloor \\
&= \Psi_{\Phi_r}(\alpha r, 1) + 2M\alpha r^2.
\end{aligned}$$

Note that in the second to last step we used the fact that when (i, j) is assigned with $-\mu_{r,j} + r\mu_F$, every admissible path in $\Pi(\lfloor \alpha n r \rfloor, n)$ contains at least one of each $-\mu_{r,j} + r\mu_F$ for every $j = 0, 1, \dots, n$. Their average converges to 0 by the law of large numbers. There are also $\lfloor \alpha n r \rfloor$ additional points all bounded from above by $2M$, which contribute to the third term.

The opposite bound $\Psi_{\tilde{\Phi}_r}(\alpha r, 1) \geq \Psi_{\Phi_r}(\alpha r, 1) - 2M\alpha r^2$ comes similarly. So

$$\frac{1}{r\sqrt{\alpha}} |\Psi_{\tilde{\Phi}_r}(\alpha r, 1) - \Psi_{\Phi_r}(\alpha r, 1)| \leq 2Mr\sqrt{\alpha} \quad (4.1.19)$$

and the lemma follows. \square

Let us separate the mean by letting $\bar{\Phi}_{r,y}$ denote the $N(0, \sum_{i=0}^{r-1} \sigma_{ry+i}^2)$ distribution function. Since the last-passage functions of the normal distributions satisfy

$$\Psi_{\tilde{\Phi}_r}(\alpha r, 1) = r\mu_F(1 + \alpha r) + \Psi_{\bar{\Phi}^{(r)}}(\alpha r, 1), \quad (4.1.20)$$

we can summarize the effect of the last three lemmas as follows.

Lemma 10. *Assume $\{F_j\}$ i.i.d. under \mathbb{P} , and assume $r = r(\alpha)$ satisfies $r \rightarrow \infty$ and $r\sqrt{\alpha} \rightarrow 0$ as $\alpha \downarrow 0$. Under assumptions (4.1.4)*

$$\lim_{\alpha \downarrow 0} \frac{1}{\sqrt{\alpha}} |\Psi_F(\alpha, 1) - \mu_F - \frac{1}{r} \Psi_{\frac{1}{\Phi(r)}}(\alpha r, 1)| = 0. \quad (4.1.21)$$

In order to deduce a limit from (4.1.21) we utilize the explicitly computable case of exponential distributions from [20], and use the results proved in Chapter 3.2. We need to match up the random variances of the exponentials with the variances σ_j^2 of the sequence $\{F_j\}$. Thus, given the i.i.d. sequence of quenched variances $\sigma_j^2 = \sigma^2(F_j)$ that we have worked with up to now under condition (4.1.5), let $\xi_j = 1/\sigma_j$ and $G_j(x) = 1 - e^{-\xi_j x}$ the rate ξ_j exponential distribution. Then $\{\xi_j\}_{j \in \mathbb{Z}_+}$ is an i.i.d. sequence of random variables $\xi_j > 0$ with distribution m . Since we assume (4.1.4), the sequence $\{\xi_j\}_{j \in \mathbb{Z}_+}$ is bounded away from zero. We can assume c is the exact lower bound: $m[c, c + \varepsilon) > 0$ for each $\varepsilon > 0$. G_j has mean and variance $\mu(G_j) = \xi_j^{-1}$ and $\sigma^2(G_j) = \xi_j^{-2} = \sigma_j^2$.

Assumptions (2.2.2) and (2.2.3) are easily checked, and so the last-passage function Ψ_G is well-defined. We would like to apply Lemma 10 to this exponential model, but obviously assumption (4.1.4) is not satisfied. To get around this difficulty we do the following approximation which leaves the quenched means and variances intact. We learned this trick from [13].

Let Y_j denote a G_j -distributed random variable. For a fixed $\tau > 0$, let

$$m_j = E(Y_j | Y_j > \tau) \quad \text{and} \quad w_j = E(Y_j^2 | Y_j > \tau).$$

The quantities

$$s_j = \frac{(m_j - \tau)^2}{(m_j - \tau)^2 + w_j - m_j^2} \quad \text{and} \quad u_j = \frac{w_j - \tau^2}{m_j - \tau} - \tau$$

satisfy the equations

$$(1 - s_j)\tau + s_j u_j = m_j \quad \text{and} \quad (1 - s_j)\tau^2 + s_j u_j^2 = w_j.$$

Then $0 \leq s_j \leq 1$, $u_j \geq \tau$ and $w_j \geq \tau^2$. Define distribution functions

$$\tilde{G}_j(x) = \begin{cases} G_j(x) & 0 \leq x < \tau \\ 1 - s_j[1 - G_j(\tau)] & \tau \leq x < u_j \\ 1 & x \geq u_j. \end{cases} \quad (4.1.22)$$

$\tilde{Y}_j \sim \tilde{G}_j$ satisfies $EY_j = E\tilde{Y}_j$ and $EY_j^2 = E\tilde{Y}_j^2$. Moreover, for any fixed $\tau > 0$,

$$u_j = \frac{E(Y_j^2 | Y_j > \tau) - \tau^2}{E(Y_j | Y_j > \tau) - \tau} - \tau = \frac{2}{p_j} + \tau \leq \frac{2}{c} + \tau,$$

so $\{\tilde{G}_j\}$ are all supported on the non-random bounded interval $[0, 2/c + \tau]$. Consequently Lemma 10 applies to \tilde{G} . We can draw the same conclusion for G once we have the next estimate:

Lemma 11. *Given $\varepsilon > 0$, we can select τ large enough and define \tilde{G}_j as in (4.1.22) so that*

$$\lim_{\alpha \downarrow 0} \frac{1}{\sqrt{\alpha}} |\Psi_G(\alpha, 1) - \Psi_{\tilde{G}}(\alpha, 1)| < \varepsilon.$$

Proof. This comes from an application of Lemma 4. $G_j = \tilde{G}_j$ on $(-\infty, \tau)$ and $1 - \tilde{G}_j \leq 1 - G_j$ on all of \mathbb{R} .

$$\begin{aligned}
& |\Psi_G(\alpha, 1) - \Psi_{\tilde{G}}(\alpha, 1)| \\
& \leq 8\sqrt{\alpha} \int_{-\infty}^{+\infty} \left(\mathbb{E}|G_0(x) - \tilde{G}_0(x)| \right)^{1/2} dx + \alpha \int_{-\infty}^{+\infty} \operatorname{ess\,sup}_{\mathbb{P}} |G_0(x) - \tilde{G}_0(x)| dx \\
& \leq 8\sqrt{\alpha} \int_{\tau}^{+\infty} \left(\mathbb{E}|1 - G_0(x)| + \mathbb{E}|1 - \tilde{G}_0(x)| \right)^{1/2} dx \\
& \quad + \alpha \int_{\tau}^{+\infty} \left(\operatorname{ess\,sup}_{\mathbb{P}} |1 - G_0(x)| + \operatorname{ess\,sup}_{\mathbb{P}} |1 - \tilde{G}_0(x)| \right) dx \\
& \leq 8\sqrt{2\alpha} \int_{\tau}^{+\infty} \left(\mathbb{E}|1 - G_0(x)| \right)^{1/2} dx + 2\alpha \int_{\tau}^{+\infty} \operatorname{ess\,sup}_{\mathbb{P}} |1 - G_0(x)| dx \\
& \leq 8\sqrt{2\alpha} \int_{\tau}^{+\infty} \exp\left(-\frac{cx}{2}\right) dx + 2\alpha \int_{\tau}^{+\infty} \exp(-cx) dx \\
& = \frac{16\sqrt{2\alpha}}{c} \exp\left(-\frac{c\tau}{2}\right) + \frac{2\alpha}{c} \exp(-c\tau).
\end{aligned} \tag{4.1.23}$$

Now we see $\lim_{\alpha \downarrow 0} \frac{1}{\sqrt{\alpha}} |\Psi_G(\alpha, 1) - \Psi_{\tilde{G}}(\alpha, 1)|$ can be made arbitrarily small by choosing τ large. \square

So Lemma 10 and Lemma 11 together show that

$$\lim_{\alpha \downarrow 0} \frac{1}{\sqrt{\alpha}} |\Psi_G(\alpha, 1) - \mathbb{E}\sigma_0 - \frac{1}{r} \Psi_{\Phi_r}(\alpha r, 1)| = 0. \tag{4.1.24}$$

It remains to perform an explicit calculation on $\Psi_G(\alpha, 1)$. As before, utilize the notation $\mu_G = \mathbb{E}\xi_0^{-1}$ and $\sigma_G^2 = \mathbb{E}\xi_0^{-2}$. In Theorem 3.2.1 We have already computed that

$$\lim_{\alpha \downarrow 0} \frac{1}{\sqrt{\alpha}} |\Psi_G(\alpha, 1) - \mu_G - 2\sigma_G\sqrt{\alpha}| = 0.$$

This result combined with (4.1.24) gives

$$\lim_{\alpha \downarrow 0} \frac{1}{\sqrt{\alpha}} \left| \frac{1}{r} \Psi_{\Phi_r}(\alpha r, 1) - 2\sigma_G \sqrt{\alpha} \right| = 0.$$

Substitute this back into (4.1.21) and recall that $\sigma_F = \sigma_G$. The conclusion we get is

$$\lim_{\alpha \downarrow 0} \frac{1}{\sqrt{\alpha}} |\Psi_F(\alpha, 1) - \mu_F - 2\sigma_F \sqrt{\alpha}| = 0. \quad (4.1.25)$$

So far we have proved Theorem 4.1.1 under the assumption (4.1.4). As the last item of the proof of Theorem 4.1.1 we remove this uniform boundedness assumption. Suppose $\{F_j\}$ satisfy the conditions required for Theorem 4.1.1, but there is no common bounded support. For a fixed $M > 0$ define the truncated distributions

$$F_{j,M}(x) = \begin{cases} 1 & x \geq M \\ F_j(x) & -M \leq x < M \\ 0 & x < -M. \end{cases}$$

Let μ_M , σ_M^2 and $\Psi_{F_M}(x, y)$ be quantities associated to $\{F_{j,M}\}$.

From (4.1.3) and the conditions assumed in Theorem 4.1.1,

$$\begin{aligned} & \frac{1}{\sqrt{\alpha}} |\Psi_F(\alpha, 1) - \Psi_{F_M}(\alpha, 1) - (\mu - \mu_M)| \\ & \leq 8 \int_{-\infty}^{+\infty} \left(\mathbb{E} |F_0(x) - F_{0,M}(x)| \right)^{1/2} dx + \sqrt{\alpha} \int_{-\infty}^{+\infty} \text{ess sup}_{\mathbb{P}} |F_0(x) - F_{0,M}(x)| dx \\ & = 8 \left[\int_{-\infty}^{-M} \left(\mathbb{E} |F_0(x)| \right)^{1/2} dx + \int_M^{\infty} \left(\mathbb{E} |1 - F_0(x)| \right)^{1/2} dx \right] \\ & \quad + \sqrt{\alpha} \left[\int_{-\infty}^{-M} \text{ess sup}_{\mathbb{P}} |F_0(x)| dx + \int_M^{+\infty} \text{ess sup}_{\mathbb{P}} |1 - F_0(x)| dx \right] \leq \varepsilon. \end{aligned} \quad (4.1.26)$$

The last inequality comes from choosing M large enough, and is valid for all $\alpha \leq 1$. Since $\mathbb{E}(EX^2(0,0)) < \infty$, dominated convergence gives $\sigma_M \rightarrow \sigma$ and so we can pick M so that $|\sigma - \sigma_M| < \varepsilon$. Now

$$\frac{1}{\sqrt{\alpha}}|\Psi_F(\alpha, 1) - \mu - 2\sigma\sqrt{\alpha}| \leq \frac{1}{\sqrt{\alpha}}|\Psi_{F_M}(\alpha, 1) - \mu_M - 2\sigma_M\sqrt{\alpha}| + 2\varepsilon.$$

Since ε is arbitrary and limit (4.1.25) holds for $\{F_{j,M}\}$, we get the conclusion for the sequence $\{F_j\}$. This concludes the proof of Theorem 4.1.1.

4.1.2 An improvement on the error $o(\sqrt{\alpha})$

The error term $o(\sqrt{\alpha})$ can still be improved. As was shown in Theorem 3.2.1, in the exponential model $O(\alpha)$ is a more accurate estimate. In this section we will take a closer look at the approximations we used along the proof of Theorem 4.1.1 and analyze the order of α .

Theorem 4.1.2. *Assume the process $\{F_j\}$ satisfies the assumptions in Theorem 4.1.1, i.e. it is i.i.d. under \mathbb{P} , and satisfies tail assumptions (2.2.2), (2.2.3), (4.1.1) and (4.1.2).*

In addition, assume the uniform bound (4.1.4), then for any $\varepsilon > 0$, as $\alpha \searrow 0$

$$\Psi_F(\alpha, 1) = \mu_F + 2\sigma_F\sqrt{\alpha} + o(\alpha^{\frac{3}{5}-\varepsilon}). \quad (4.1.27)$$

Proof. We give a list of all approximations in the previous subsection: when $\{F_j\}_{j \in \mathbb{Z}_+}$

satisfies (4.1.4),

$$(4.1.6) : \frac{1}{\sqrt{\alpha}} |\Psi(\alpha, 1) - \frac{1}{r} \Psi(\alpha r, 1)| \leq Mr\sqrt{\alpha},$$

$$(4.1.16) : \frac{1}{r\sqrt{\alpha}} |\Psi_{F_r}(\alpha r, 1) - \Psi_{\Phi_r}(\alpha r, 1)| \leq C(M^{(5+\delta)/2} r^{(-1+\delta)/4} + Mr\sqrt{\alpha}),$$

$$(4.1.19) : \frac{1}{r\sqrt{\alpha}} |\Psi_{\tilde{\Phi}_r}(\alpha r, 1) - \Psi_{\Phi_r}(\alpha r, 1)| \leq 2Mr\sqrt{\alpha}, \text{ and}$$

$$(4.1.20) : \Psi_{\tilde{\Phi}_r}(\alpha r, 1) = r\mu_F(1 + \alpha r) + \Psi_{\tilde{\Phi}(r)}(\alpha r, 1).$$

To sum up, (4.1.21) can be rewritten as

$$|\Psi_F(\alpha, 1) - \mu_F - \frac{1}{r} \Psi_{\tilde{\Phi}(r)}(\alpha r, 1)| \leq C(M^{(5+\delta)/2} r^{(-1+\delta)/4} \sqrt{\alpha} + Mr\alpha) \quad (4.1.28)$$

for a proper constant C .

Now we recall the approximation for the exponential model. For exponential distributions $\{G_j\}_{j \in \mathbb{Z}_+}$, if we define $\{\tilde{G}_j\}_{j \in \mathbb{Z}_+}$ uniformly bounded by M as in (4.1.22), then (4.1.23) gives:

$$|\Psi_G(\alpha, 1) - \Psi_{\tilde{G}}(\alpha, 1)| \leq C \exp(-cM) \sqrt{\alpha}$$

for a proper constant C .

The above equation, together with (4.1.28) applied to $\{\tilde{G}_j\}_{j \in \mathbb{Z}_+}$, implies

$$|\Psi_G(\alpha, 1) - \mu_G - \frac{1}{r} \Psi_{\tilde{\Phi}(r)}(\alpha r, 1)| \leq C(\exp(-cM) \sqrt{\alpha} + M^{(5+\delta)/2} r^{(-1+\delta)/4} \sqrt{\alpha} + Mr\alpha). \quad (4.1.29)$$

Here C does not depend on M, r , and α .

In the above equation the left-hand side is independent of M , so we make both M and r functions of α . Let $r(\alpha) = \alpha^{-\frac{2}{5-\delta}}$ and $M(\alpha) = -\frac{0.1}{c} \ln \alpha$. Make δ small enough, then the right hand side of (4.1.29) is bounded by $C(-\frac{0.1}{c} \ln \alpha)^{5+\delta} \alpha^{\frac{3-\delta}{5-\delta}} = o(\alpha^{\frac{3}{5}-\varepsilon})$ as $\alpha \searrow 0$.

Recall Theorem 3.2.1, we get

$$\frac{1}{r} \Psi_{\Phi(r)}(\alpha r, 1) = 2\sigma_G \sqrt{\alpha} + o(\alpha^{\frac{3}{5}-\varepsilon}). \quad (4.1.30)$$

Now let us come back to $\Psi(\alpha, 1)$. If $\{F_j\}_{j \in \mathbb{Z}_+}$ are uniformly bounded, then M is a fixed constant and the right hand side of (4.1.28) is just $o(\alpha^{\frac{3}{5}-\varepsilon})$, and this shows that $\forall \varepsilon > 0$,

$$\Psi(\alpha, 1) = \mu_F + 2\sigma_F \sqrt{\alpha} + o(\alpha^{\frac{3}{5}-\varepsilon}).$$

□

Remark 12. *The proof did not treat the case when $\{F_j\}_{j \in \mathbb{Z}_+}$ is not uniformly bounded. The difficulty is that (4.1.26) does not give precise computability on how the right-hand side would change according to M . We surely need additional assumptions in this case.*

4.2 Estimates for limiting shape near x -axis

We turn to the case $\Psi(1, \alpha)$. As we have seen in 3.2.3, the results will be qualitatively different from Theorem 4.1.1. The leading term will be the essential supremum of the mean instead of the averaged mean and we will see different orders for the first α -dependent correction term. Universality results are desired but much more difficult to achieve. In this section we will see some estimates of $\Psi(1, \alpha)$ under various conditions.

We first present a result which gives an upper bound of $\Psi(1, \alpha)$ in a very general setting. We will use F as superscripts or subscripts when we want to stress the dependence (of a probability, or expectation, etc.) on a distribution function F that is in the state space of \mathbb{P} . Again, $\mu(F)$ is the mean of the distribution F and $\sigma^2(F)$ is the variance.

Theorem 4.2.1. *Assume $\{F_j\}_{j \in \mathbb{Z}_+}$ is i.i.d. and there exists constants $t_0 > 0$ and $K_0 < \infty$ such that*

$$\text{ess sup}_{\mathbb{P}} E^F e^{t_0 |X - \mu(F)|} < K_0.$$

Let $\mu^ = \text{ess sup}_{\mathbb{P}} \mu(F)$, then*

$$\Psi(1, \alpha) = \mu^* + O\left(\sqrt{\alpha \log \frac{1}{\alpha}}\right). \quad (4.2.1)$$

Proof. We will use ω to denote a realization of $\{F_j\}_{j \in \mathbb{Z}_+}$, and use P^ω and E^ω as the corresponding quenched probability and expectation. Recall the definition

$$\Psi(1, \alpha) = \lim_{n \rightarrow \infty} \frac{1}{n} \max_{\pi \in \Pi(n, \lfloor n\alpha \rfloor)} \sum_{z \in \pi} X(z).$$

For any specific path π , we write $T(\pi) = \sum_{z \in \pi} X(z)$. We also use m_j as the leftmost site traveled by π in the j -th row. Then we derive the following estimate: for any positive u and t ,

$$\begin{aligned}
\mathbf{P}(T(\pi) \geq nu) &= \mathbb{E}P^\omega(T(\pi) \geq nu) \leq \mathbb{E}(e^{-ntu} E^\omega(e^{tT(\pi)})) \\
&= e^{-ntu} \mathbb{E}[\exp(\sum_{z \in \pi} \log E^\omega(e^{tX(z)}))] \\
&= e^{-ntu} \prod_{j=0}^{\lfloor n\alpha \rfloor} \mathbb{E} \exp\left[\sum_{i=m_j}^{m_{j+1}} \log E^\omega(e^{tX(i,j)})\right] \\
&= \exp\left\{-ntu + \sum_{j=0}^{\lfloor n\alpha \rfloor} \log \mathbb{E}[\exp((1 + m_{j+1} - m_j) \log E^{F_j} e^{tX})]\right\}.
\end{aligned} \tag{4.2.2}$$

We need the following inequality in order to proceed: if we have a random variable Y and positive real numbers t_1, \dots, t_m with $\sum_j t_j = t$, then

$$\sum_j \log E e^{t_j Y} \leq \sum_j \log [E(e^{t_j Y})]^{\frac{t_j}{t}} = \sum_j \frac{t_j}{t} \log [E(e^{t_j Y})] = \log [E(e^{tY})]. \tag{4.2.3}$$

Continuing from (4.2.2), as $n \nearrow \infty$

$$\begin{aligned}
\mathbf{P}(T(\pi) \geq nu) &\leq \exp\left\{-ntu + \log \mathbb{E} \exp(n(1 + \alpha) \log E^F e^{tX})\right\} \\
&= \exp\left\{-n[tu - (1 + \alpha) \frac{1}{n(1 + \alpha)} \log \mathbb{E} \exp(n(1 + \alpha) \log E^F e^{tX})]\right\} \\
&\leq \exp\left\{-n[tu - (1 + \alpha) \Lambda(t)]\right\},
\end{aligned} \tag{4.2.4}$$

where $\Lambda(t) = \text{ess sup}_{\mathbb{P}} \Lambda^F(t)$ and $\Lambda^F(t) = \log E^F e^{tX}$.

From the assumption that there is $t_0 > 0$ and $K_0 < \infty$ such that $E^F e^{t_0 |X - \mu(F)|} < K_0$,

it follows that for $t \in (0, t_0)$,

$$\begin{aligned}
\Lambda^F(t) &= \mu(F)t + \log\left(1 + \frac{\sigma^2(F)}{2}t^2 + E^F \sum_{k=3}^{\infty} \frac{t^k}{k!} (X - \mu(F))^k\right) \\
&\leq \mu(F)t + \frac{\sigma^2(F)}{2}t^2 + t^3 E^F e^{t_0|X - \mu(F)|} \\
&\leq \mu(F)t + \frac{\sigma^2(F)}{2}t^2 + K_0 t^3,
\end{aligned} \tag{4.2.5}$$

and therefore if we denote $\sigma^* = \text{ess sup}_{\mathbb{P}} \sigma(F)$,

$$\Lambda(t) \leq \mu^* t + \frac{\sigma^{*2}}{2} t^2 + K_0 t^3,$$

from which we obtain

$$\begin{aligned}
\mathbf{P}(T(\pi) \geq nu) &\leq \exp\left\{-n[tu - (1 + \alpha)(\mu^* t + \frac{\sigma^{*2}}{2}t^2 + K_0 t^3)]\right\} \\
&= \exp\left\{-n[(u - (1 + \alpha)\mu^*)t - (1 + \alpha)\frac{\sigma^{*2}}{2}t^2 - (1 + \alpha)K_0 t^3]\right\}.
\end{aligned} \tag{4.2.6}$$

We take $u = \mu^*(1 + \alpha) + \varepsilon$ and $t = \frac{\varepsilon}{\sigma^{*2}(1 + \alpha)}$, where $\varepsilon = 4\sigma^* \sqrt{\alpha \log \frac{1}{\alpha}}$. When α is small enough, ε is small and we get $t \in (0, t_0)$. Then (4.2.6) becomes

$$\mathbf{P}(T(\pi) \geq nu) \leq \exp\left[-n \frac{\varepsilon^2}{2\sigma^{*2}(1 + \alpha)} \left(1 - \frac{2K_0\varepsilon}{\sigma^{*4}(1 + \alpha)}\right)\right] \leq \exp\left[-n \frac{\varepsilon^2}{4\sigma^{*2}(1 + \alpha)}\right] \tag{4.2.7}$$

when α and thus ε is small enough.

The last-passage time $T(n, \lfloor n\alpha \rfloor) = \max_{\pi \in \Pi(n, \lfloor n\alpha \rfloor)} T(\pi)$. Here the maximum is taken

over a pool of $\binom{n+\lfloor n\alpha \rfloor}{n}$ paths, so we can use Stirling's formula to get

$$\begin{aligned} \mathbf{P}(T(n, \lfloor n\alpha \rfloor) \geq nu) &\leq \binom{n+\lfloor n\alpha \rfloor}{n} \exp\left[-n \frac{\varepsilon^2}{4\sigma^{*2}(1+\alpha)}\right] \\ &\leq \frac{C}{\sqrt{\alpha}} \exp\left[-n\left(\alpha \log \alpha - (1+\alpha) \log(1+\alpha) + \frac{\varepsilon^2}{4\sigma^{*2}(1+\alpha)}\right)\right]. \end{aligned} \quad (4.2.8)$$

Plug in the expression for ε we have

$$\alpha \log \alpha - (1+\alpha) \log(1+\alpha) + \frac{\varepsilon^2}{4\sigma^{*2}(1+\alpha)} = \frac{3-\alpha}{1+\alpha} \alpha \log \frac{1}{\alpha} - (1+\alpha) \log(1+\alpha).$$

As $\alpha \searrow 0$, the first term on the right-hand side above has order $\alpha \log \frac{1}{\alpha}$, whereas the second term has order α . So when α is small,

$$\alpha \log \alpha - (1+\alpha) \log(1+\alpha) + \frac{\varepsilon^2}{4\sigma^{*2}(1+\alpha)} > 0$$

and hence

$$\sum_n \mathbf{P}(T(n, \lfloor n\alpha \rfloor) \geq nu) < \infty,$$

which by Borel-Cantelli lemma gives

$$\Psi(1, \alpha) \leq u = \mu^*(1+\alpha) + 4\sigma^* \sqrt{\alpha \log \frac{1}{\alpha}}, \quad (4.2.9)$$

and we can claim as $\alpha \searrow 0$,

$$\Psi(1, \alpha) = \mu^* + O\left(\sqrt{\alpha \log \frac{1}{\alpha}}\right). \quad (4.2.10)$$

□

The order $\sqrt{\alpha \log \frac{1}{\alpha}}$ should be a rather conservative estimate. In Theorem 3.2.3 and in some other settings we find that the first α -dependent term in $\Psi(1, \alpha)$ is no more than $\sqrt{\alpha}$. There has not been a very general statement about the necessary condition for the order $\sqrt{\alpha}$ so far, but next we will see two sufficient conditions. The first one is uniform boundedness.

Theorem 4.2.2. *Let $\{F_j\}_{j \in \mathbb{Z}_+}$ be an ergodic sequence of distribution functions satisfying the conditions listed in Proposition 2.1. Assume the existence of $M > 0$ such that*

$$\mathbb{P}\{F_0(-M) = 0, F_0(M) = 1\} = 1. \quad (4.2.11)$$

Again define $\mu^ = \text{ess sup}_{\mathbb{P}} \mu(F)$, then as $\alpha \searrow 0$,*

$$\Psi(1, \alpha) = \mu^* + O(\sqrt{\alpha}). \quad (4.2.12)$$

Proof. We first prove an upper bound for $\Psi(1, \alpha)$. We start by increasing all the weights $X(z)$ by moving their means to μ^* , so that $\Psi(1, \alpha)$ for the shifted weights gets no smaller. Then we subtract the common mean μ^* from the weights. Therefore we can assume $\mu(F) = 0$ for all F . The weights $X(z)$ are still uniformly bounded, and without loss of generality we still assume (4.2.11) for the shifted weights with the bounds still denoted by M .

Fix a realization of $\{F_j\}_{j \in \mathbb{Z}_+}$, and the lattice point $z_0 = (0, 0)$. Let N be a positive integer. According to whether the path π goes through z_0 or not, and in case it goes we

also separate the weight at z_0 , we write

$$\max_{\pi \in \Pi(n, \lfloor N\alpha \rfloor)} \sum_{z \in \pi} X(z) = A \vee (B + X(z_0)) = B + (A - B) \vee X(z_0), \quad (4.2.13)$$

where $A = \max_{z_0 \notin \pi} \sum_{z \in \pi} X(z)$ and $B = \max_{z_0 \in \pi} \sum_{z \in \pi \setminus \{z_0\}} X(z)$. Both A and B look complicated but we only need to treat them as some random variables. Let $G(y)$ denote the distribution of $A - B$, then the quenched expectation

$$E[(A - B) \vee X(z_0)] = \int_{\mathbb{R} \times \mathbb{R}} x \vee y dF_0(x) dG(y) = \int_{\mathbb{R}} \left(\int_{-M}^M x \vee y dF_0(x) \right) dG(y). \quad (4.2.14)$$

We now take a closer look at $\int_{-M}^M x \vee y dF_0(x)$. The only nontrivial case is when $y \in [-M, M]$, integration by parts gives

$$\begin{aligned} \int_{-M}^M x \vee y dF_0(x) &= yF_0(y) + \int_y^M x dF_0(x) \\ &= yF_0(y) + (M - yF_0(y)) - \int_y^M F_0(x) dx \\ &= M - \int_y^M F_0(x) dx. \end{aligned} \quad (4.2.15)$$

Next we try to minimize the integral $\int_y^M F(x) dx$ when $F(x)$ is selected from mean zero distribution functions supported on $[-M, M]$. Suppose the value $F(y)$ is given and $F(y) \geq \frac{1}{2}$, then obviously $\int_y^M F(x) dx \geq (M - y)F(y)$, in which the equal sign can be

achieved for

$$F(x) = \begin{cases} 0 & x < -M \\ \frac{(1-F(y))M+yF(y)}{M+y} & -M \leq x < y \\ F(y) & y \leq x < M \\ 1 & x \geq M. \end{cases} \quad (4.2.16)$$

If $F(y)$ is known and $F(y) < \frac{1}{2}$, the above function $F(x)$ would not work since it is then not non-decreasing. Now we have

$$\begin{aligned} \int_y^M F(x)dx &= \int_{-M}^M F(x)dx - \int_{-M}^y F(x)dx \\ &= M - \int_{-M}^M x dF(x) - \int_{-M}^y F(x)dx \\ &= M - EX - \int_{-M}^y F(x)dx \\ &\geq M - F(y)(y + M). \end{aligned} \quad (4.2.17)$$

The equality holds when

$$F(x) = \begin{cases} 0 & x < -M \\ F(y) & -M \leq x < y \\ \frac{(1-F(y))M-yF(y)}{M-y} & y \leq x < M \\ 1 & x \geq M. \end{cases} \quad (4.2.18)$$

We now summarize the above two cases and see $\int_y^M F(x)dx \geq \frac{1}{2}(M-y)$, with equality

when $F(y) = \frac{1}{2}$, which corresponds to the distribution function

$$F^M(x) = \begin{cases} 0 & x < -M \\ \frac{1}{2} & -M \leq x < M \\ 1 & M \leq x. \end{cases} \quad (4.2.19)$$

Notice that $F^M(x)$ puts half probability on M and $-M$ each and does not depend on the value y . Back to (4.2.14), we see that for any random variables A and B , $E[(A - B) \vee X(z_0)]$ is maximized as long as we let $X(z_0)$ follow $F^M(x)$. Running this argument for all $z \in \{0, \dots, N\} \times \{0, \dots, \lfloor N\alpha \rfloor\}$, we obtain

$$E \max_{\pi \in \Pi(n, \lfloor N\alpha \rfloor)} \sum_{z \in \pi} X(z) \leq E \max_{\pi \in \Pi(n, \lfloor N\alpha \rfloor)} \sum_{z \in \pi} X_{F^M}(z). \quad (4.2.20)$$

Taking limits and using (1.0.3) gives

$$\Psi(1, \alpha) \leq \Psi_{F^M}(1, \alpha) = 2M\sqrt{\alpha} + o(\sqrt{\alpha}). \quad (4.2.21)$$

If we consider the effect of μ^* , we get (4.2.12).

□

The next two theorems also prove that $\Psi(1, \alpha)$ is a constant plus order $O(\sqrt{\alpha})$. They relax the assumption of uniform boundedness and use the ones from [13]. The finiteness of the state space of \mathbb{P} plays an important role in the proofs of both theorems, but it does not seem to be a necessary condition for the results. It would be great if we can design a different approach and remove this finiteness condition.

Theorem 4.2.3. *Assume the process $\{F_j\}$ of probability distributions is stationary, ergodic, and has a state space of finitely many distributions H_1, \dots, H_L each of which satisfies Martin's [13] hypothesis*

$$\int_0^\infty (1 - H_\ell(x))^{1/2} dx + \int_{-\infty}^0 H_\ell(x)^{1/2} dx < \infty. \quad (4.2.22)$$

Let $\mu^* = \max_\ell \mu(H_\ell)$ be the maximal mean of the H_ℓ 's. Then there exist constants $0 < c_1 < c_2 < \infty$ such that, as $\alpha \downarrow 0$,

$$\mu^* + c_1\sqrt{\alpha} + o(\sqrt{\alpha}) \leq \Psi(1, \alpha) \leq \mu^* + c_2\sqrt{\alpha} + o(\sqrt{\alpha}). \quad (4.2.23)$$

Proof. The lower bound in (4.2.23) can be proved by applying Martin's result (1.0.3) to the homogeneous problem where a maximal path is constructed by using only those rows j where $F_j = H_{i^*}$, the distribution with the maximal mean $\mu^* = \mu(H_{i^*})$. This is fairly straightforward.

To prove the upper bound in (4.2.23), we again start by increasing all the weights $X(z)$ by moving their means to μ^* . Then we subtract the common mean μ^* from the weights, so that for the proof we can assume that all distributions H_1, \dots, H_L have mean zero.

Create the following coupling. Independently of the process $\{F_j\}$, let $\{X_\ell(z) : 1 \leq \ell \leq L, z \in \mathbb{Z}_+^2\}$ be a collection of independent weights such that $X_\ell(z)$ has distribution

H_ℓ . Then define the weights used for computing $\Psi(1, \alpha)$ by

$$X(z) = \sum_{\ell=1}^L X_\ell(z) I_{\{F_j=H_\ell\}} \quad \text{for } z = (i, j) \in \mathbb{Z}_+^2.$$

Begin with this elementary bound:

$$\begin{aligned} \Psi(1, \alpha) &= \lim_{n \rightarrow \infty} \frac{1}{n} \mathbf{E} \left[\max_{\pi \in \Pi(n, \lfloor \alpha n \rfloor)} \sum_{z \in \pi} X(z) \right] \\ &\leq \sum_{\ell=1}^L \lim_{n \rightarrow \infty} \frac{1}{n} \mathbf{E} \left[\max_{\pi \in \Pi(n, \lfloor \alpha n \rfloor)} \sum_{z \in \pi} X_\ell(z) I_{\{F_j=H_\ell\}} \right]. \end{aligned} \tag{4.2.24}$$

The next lemma contains a convexity argument that will remove the indicators from the last-passage values above.

Lemma 13. *Let \mathcal{D} be a sub- σ -field on a probability space (Ω, \mathcal{F}, P) , D an event in \mathcal{D} , and ξ and η two integrable random variables. Assume that $E\eta = 0$, η is independent of \mathcal{D} , and ξ and η are independent conditionally on \mathcal{D} . Then $E[\xi \vee (\eta I_D)] \leq E[\xi \vee \eta]$.*

Proof. By Jensen's inequality, for any fixed $x \in \mathbb{R}$,

$$x \vee E(\eta | \mathcal{D}) \leq E(x \vee \eta | \mathcal{D}).$$

Since η is independent of \mathcal{D} and mean zero,

$$x \vee 0 \leq E(x \vee \eta | \mathcal{D}).$$

Integrate this against the conditional distribution $P(\xi \in dx | \mathcal{D})$ of ξ , given \mathcal{D} , and use

the conditional independence of ξ and η :

$$E(\xi \vee 0 \mid \mathcal{D}) \leq E(\xi \vee \eta \mid \mathcal{D}).$$

Next integrate this over the event D^c :

$$E[I_{D^c} \cdot \xi \vee (\eta I_D)] = E[I_{D^c} \cdot \xi \vee 0] \leq E[I_{D^c} \cdot \xi \vee \eta].$$

The corresponding integral over the event D needs no argument. \square

Fix a lattice point $z_0 = (i_0, j_0)$ for the moment. We split the maximum in (4.2.24) like the way we did in (4.2.13):

$$\max_{\pi \in \Pi(n, \lfloor n\alpha \rfloor)} \sum_{z \in \pi} X_\ell(z) I_{\{F_j = H_\ell\}} = B + (A - B) \vee (X_\ell(z_0) I_{\{F_{j_0} = H_\ell\}})$$

where

$$A = \max_{\pi \not\ni z_0} \sum_{z \in \pi} X_\ell(z) I_{\{F_j = H_\ell\}} \quad \text{and} \quad B = \max_{\pi \ni z_0} \sum_{z \in \pi \setminus \{z_0\}} X_\ell(z) I_{\{F_j = H_\ell\}}.$$

Now apply Lemma 13 with $\xi = A - B$, $\eta = X_\ell(z_0)$, and $D = \{F_{j_0} = H_\ell\}$. Given F_{j_0} , $A - B$ does not look at $X_\ell(z_0)$, so the independence assumed in Lemma 13 is satisfied.

The outcome from that lemma is the inequality

$$\mathbf{E} \left[\max_{\pi \in \Pi(n, \lfloor \alpha n \rfloor)} \sum_{z \in \pi} X_\ell(z) I_{\{F_j = H_\ell\}} \right] \leq \mathbf{E} [A \vee (B + X_\ell(z_0))].$$

This is tantamount to replacing the weight $X_\ell(z_0) I_{\{F_{j_0} = H_\ell\}}$ at z_0 with $X_\ell(z_0)$.

We can repeat this at all lattice points z_0 in (4.2.24). In the end we have an upper

bound in terms of homogeneous last-passage values, to which we can apply Martin's result (1.0.3):

$$\begin{aligned}\Psi(1, \alpha) &\leq \sum_{\ell=1}^L \lim_{n \rightarrow \infty} \frac{1}{n} \mathbf{E} \left[\max_{\pi \in \Pi(n, [\alpha n])} \sum_{z \in \pi} X_{\ell}(z) \right] = \sum_{\ell=1}^L \Psi_{H_{\ell}}(1, \alpha) \\ &= 2\sqrt{\alpha} \sum_{\ell=1}^L \sigma(H_{\ell}) + o(\sqrt{\alpha}).\end{aligned}$$

This completes the proof of Theorem 4.2.3. \square

The gist of the above theorem is the inequality

$$\Psi(1, \alpha) \leq \sum_{\ell=1}^L \Psi_{H_{\ell}}(1, \alpha). \quad (4.2.25)$$

We will use it repeatedly in the proof of the following theorem.

The following theorem also shows an order of $\sqrt{\alpha}$ given that the state space of \mathbb{P} is finite. It uses a similar approach as in the proof of Theorem 4.1.1 and is much lengthier than the previous result, but it gives a better coefficient of $\sqrt{\alpha}$ in the sense that it does not depend on the size L . Therefore it gives some insight on the possibility to remove the finiteness condition.

Theorem 4.2.4. *Let $\{F_j\}_{j \in \mathbb{Z}_+}$ be an i.i.d. sequence under \mathbb{P} from a finite set of distributions $\{H_1, \dots, H_L\}$. Again assume for each ℓ ,*

$$\int_{-\infty}^0 H_{\ell}(x)^{1/2} dx + \int_0^{\infty} (1 - H_{\ell}(x))^{1/2} dx < \infty. \quad (4.2.26)$$

Then as $\alpha \searrow 0$:

$$\Psi(1, \alpha) \leq \mu^* + 2\sigma^* \sqrt{\alpha} + o(\sqrt{\alpha}), \quad (4.2.27)$$

where $\mu^* = \max_\ell \{\mu(H_\ell)\}$ and $\sigma^* = \sqrt{\max_\ell \{\sigma^2(H_\ell)\}}$.

Proof. Again if we assume $\mu(H_\ell) = \mu^*$ for all $\ell = 1, \dots, L$, $\Psi(1, \alpha)$ will get no smaller. Without loss of generality, we will assume $\mu(H_\ell) \equiv 0$ hereafter unless specified otherwise.

In a similar way as we did in the proof of Theorem 4.1.1, we let $r = r(\alpha)$ be a positive integer-valued function such that $r(\alpha) \nearrow \infty$ and $r\sqrt{\alpha} \searrow 0$ as $\alpha \searrow 0$. Define the $r \times 1$ blocks as $B_r(x, y) = \{(rx + i, y) : i = 0, 1, \dots, r - 1\}$ for $(x, y) \in \mathbb{Z}_+^2$.

For every point $z = (i, j) \in \mathbb{Z}_+^2$, write $X_r(z) = \sum_{v \in B_r(z)} X(v)$. The distribution function of $X_r(z)$ is denoted as $F_{r,j}(x)$, and the corresponding last-passage time function for $X_r(z)$ is then denoted as $\Psi_r(x, y)$. Also, write H_r^ℓ for the convolution $H_\ell * H_\ell * \dots * H_\ell$ with H_ℓ repeated r times.

For $j \in \mathbb{Z}_+$, let $\Phi_{r,j}$ be the distribution function of the normal distribution $\mathcal{N}(0, r\sigma_j^2)$, where $\sigma_j^2 = \text{Var}(F_j)$. For each $z = (i, j) \in \mathbb{Z}_+^2$, let $Y_r(z)$ be a random variable with distribution $\Phi_{r,j}$. Write

$$\Psi_{\Phi_r}(x, y) = \lim_{n \rightarrow \infty} \frac{1}{n} \mathbf{E} \max_{\pi \in \Pi([nx], [ny])} \sum_{z \in \pi} Y_r(z).$$

Also, let Φ_r^ℓ be the distribution function of $\mathcal{N}(0, rV_\ell)$, where V_ℓ is the variance of H_ℓ .

We make an approximation first:

Lemma 14.

$$\lim_{\alpha \downarrow 0} \frac{1}{\sqrt{\alpha}} |\Psi(1, \alpha) - \frac{1}{r} \Psi_{\Phi_r}(1, r\alpha)| = 0. \quad (4.2.28)$$

Proof. For each $\ell = 0, 1, \dots, L$ and each $z = (i, j) \in \mathbb{Z}_+^2$, we do the following coupling. Define $\{u(z) : z = (i, j) \in \mathbb{Z}_+^2\}$ be i.i.d. Uniform(0, 1) random variables. Set $X_r(z) = F_{r,j}^{-1}(u(z))$, where $F_{r,j}^{-1}(u) = \sup\{x : F_{r,j}(x) < u\}$. Similarly define $Y_r(z) =$

$\Phi_{r,j}^{-1}(u(z))$. Also define $X_{r,\ell}(z) = F_{r,j}^{-1}(u(z))I_{\{F_j=H_\ell\}}$ and $Y_{r,\ell}(z) = \Phi_{r,j}^{-1}(u(z))I_{\{F_j=H_\ell\}}$.

We first assume $\{H_1, H_2, \dots, H_L\}$ are uniformly bounded and directly quote the computation from Lemma 4.2 and Lemma 4.5 in [13]:

$$\begin{aligned}
& \lim_{n \rightarrow \infty} \frac{1}{n} E \max_{\pi \in \Pi(n, \lfloor nr\alpha \rfloor)} \sum_{z \in \pi} ((H_r^\ell)^{-1}(u(z)) - (\Phi_r^\ell)^{-1}(u(z))) \\
& \leq 2\sqrt{r\alpha(1+r\alpha)} \int_{-\infty}^{\infty} |H_r^\ell(s) - \Phi_r^\ell(s)|^{\frac{1}{2}} ds \\
& \leq 2\sqrt{r\alpha(1+r\alpha)} \int_{-\infty}^{\infty} \sqrt{Cr^{-\frac{1}{2}}(1 + |\frac{s}{\sqrt{r}\sigma_\ell}|)^{-3}} ds \\
& \leq Cr^{\frac{3}{4}}\sigma_\ell\sqrt{\alpha(1+r\alpha)}
\end{aligned} \tag{4.2.29}$$

for proper constant C that are independent of r , ℓ and α .

Then we apply (4.2.25):

$$\begin{aligned}
& \Psi_r(1, r\alpha) - \Psi_{\Phi_r}(1, r\alpha) \\
& \leq \overline{\lim}_{n \rightarrow \infty} \frac{1}{n} \mathbf{E} \max_{\pi \in \Pi(n, \lfloor nr\alpha \rfloor)} \sum_{z \in \pi} (X_r(z) - Y_r(z)) \\
& \leq \sum_{\ell=0}^L \overline{\lim}_{n \rightarrow \infty} \frac{1}{n} \mathbf{E} \max_{\pi \in \Pi(n, \lfloor nr\alpha \rfloor)} \sum_{z \in \pi} (X_{r,\ell}(z) - Y_{r,\ell}(z)) \\
& \leq \lim_{n \rightarrow \infty} \frac{1}{n} \mathbf{E} \max_{\pi \in \Pi(n, \lfloor nr\alpha \rfloor)} \sum_{z \in \pi} ((H_r^\ell)^{-1}(u(z)) - (\Phi_r^\ell)^{-1}(u(z))) \\
& = \sum_{\ell=0}^L Cr^{\frac{3}{4}}\sigma_\ell\sqrt{\alpha(1+r\alpha)}.
\end{aligned} \tag{4.2.30}$$

If we switch the two terms on the left hand side, we can repeat the calculation and obtain

$$\frac{1}{r\sqrt{\alpha}} |\Psi_r(1, r\alpha) - \Psi_{\Phi_r}(1, r\alpha)| \leq \sum_{k=0}^K C\sigma_\ell r^{-\frac{1}{4}} \sqrt{1+r\alpha}, \tag{4.2.31}$$

which goes to 0 as $\alpha \searrow 0$.

Lemma 4.4 in [13] still applies here since $X(z)$'s are uniformly bounded, so we have

$$\lim_{\alpha \downarrow 0} \frac{1}{\sqrt{\alpha}} |\Psi(1, \alpha) - \frac{1}{r} \Psi_r(1, r\alpha)| = 0.$$

Therefore

$$\lim_{\alpha \downarrow 0} \frac{1}{\sqrt{\alpha}} |\Psi(1, \alpha) - \frac{1}{r} \Psi_{\Phi_r}(1, r\alpha)| = 0. \quad (4.2.32)$$

If we do not have uniform boundedness, Lemma 4.3 of [13] says that for any $\varepsilon > 0$, we can find distribution functions \tilde{H}_ℓ for each k with bounded support, and with mean and variance equal to those of H_ℓ , and $\int_{-\infty}^{\infty} |\tilde{H}_\ell(s) - H_\ell(s)|^{\frac{1}{2}} ds < \varepsilon$. Hence (4.2.32) holds for $\tilde{\Psi}(1, \alpha)$, the corresponding last-passage time function associated with $\{\tilde{H}_1, \dots, \tilde{H}_L\}$.

We repeat the argument leading to (4.2.29) and (4.2.30) and get

$$\begin{aligned} & |\tilde{\Psi}(1, \alpha) - \Psi(1, \alpha)| \\ & \leq \sum_{\ell=1}^L 2\sqrt{\alpha(1+\alpha)} \int_{-\infty}^{\infty} |\tilde{H}_\ell(s) - H_\ell(s)|^{\frac{1}{2}} ds \\ & \leq 2L\varepsilon\sqrt{\alpha(1+\alpha)}. \end{aligned} \quad (4.2.33)$$

We let ε approach 0 and it follows that (4.2.32) is also valid for $\{H_\ell\}$. The lemma is proved. \square

With Lemma 4.2.32, we know that $\Psi(1, \alpha)$ has the same coefficient of the term $\sqrt{\alpha}$ with $\frac{1}{r} \Psi_{\Phi_r}(1, r\alpha)$. $\Psi_{\Phi_r}(1, r\alpha)$ is the last-passage constant of a model where all $X(z)$ follow mean zero normal distributions. The following lemma looks at the role played by the variances of normal distributions.

Lemma 15. *Let X and Y be independent random variables and $X \sim \mathcal{N}(0, \sigma^2)$, then*

$E(X \vee Y)$ is an increasing function of σ .

Proof. Firstly, we have

$$E(X \vee Y) = \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^y y \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{x^2}{2\sigma^2}\right) dx + \int_y^{\infty} x \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{x^2}{2\sigma^2}\right) dx \right\} dG(y).$$

Note that

$$\int_{-\infty}^y x \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{x^2}{2\sigma^2}\right) dx + \int_y^{\infty} x \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{x^2}{2\sigma^2}\right) dx = 0,$$

we get

$$\begin{aligned} & \int_{-\infty}^y y \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{x^2}{2\sigma^2}\right) dx + \int_y^{\infty} x \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{x^2}{2\sigma^2}\right) dx \\ &= \int_{-\infty}^y y \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{x^2}{2\sigma^2}\right) dx - \int_{-\infty}^y x \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{x^2}{2\sigma^2}\right) dx \\ &= \int_{-\infty}^y (y - x) \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{x^2}{2\sigma^2}\right) dx \\ &= \int_{-\infty}^{\frac{y}{\sigma}} (y - \sigma x) \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx. \end{aligned}$$

The last line can be viewed as a function of σ and its derivative is

$$-\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{y}{\sigma}} x \exp\left(-\frac{x^2}{2}\right) dx > 0,$$

because when $y \leq 0$, it is easy to see $\int_{-\infty}^{\frac{y}{\sigma}} x \exp\left(-\frac{x^2}{2}\right) dx < 0$; when $y > 0$,

$$\int_{-\infty}^{\frac{y}{\sigma}} x \exp\left(-\frac{x^2}{2}\right) dx < \int_{-\infty}^{\infty} x \exp\left(-\frac{x^2}{2}\right) dx = 0.$$

Therefore $E(X \vee Y)$ is an increasing function of σ . □

With Lemma 15, we can run a similar argument with the one leading to (4.2.20). That is, we fix a $z_0 \in \mathbb{Z}_+^2$, and claim that $E \max_{\pi \in \Pi(n, \lfloor n\alpha \rfloor)} \sum_{z \in \pi} Y_r(z)$ is maximized when $Y_r(z_0)$ has the largest possible standard variance $r\sigma^{*2}$. Repeat this reasoning we see that an upper bound for $\Psi_{\Phi_r}(1, r\alpha)$ is given if we let all sites $z \in \mathbb{Z}_+^2$ have the largest possible variance. So (1.0.3) can be applied here: as $\alpha \searrow 0$,

$$\Psi_{\Phi_r}(1, r\alpha) \leq 2\sqrt{r}\sigma^* \sqrt{r\alpha} + o(\sqrt{r\alpha}).$$

From 4.2.32, as $\alpha \searrow 0$, we have

$$\Psi(1, \alpha) \leq 2\sigma^* \sqrt{\alpha} + o(\sqrt{\alpha}). \quad (4.2.34)$$

Consider the case $\mu^* \neq 0$, the last result becomes

$$\Psi(1, \alpha) \leq \mu^* + 2\sigma^* \sqrt{\alpha} + o(\sqrt{\alpha}). \quad (4.2.35)$$

□

Remark 16. *This theorem did not remove the finiteness of the state space because the approximations (4.2.30) and (4.2.33) depend on the size L . A more accurate method of approximation is needed in order to lift this assumption.*

Bibliography

- [1] E. D. Andjel, P. A. Ferrari, H. Guiol, and C. Landim. Convergence to the maximal invariant measure for a zero-range process with random rates. *Stochastic Process. Appl.*, 90(1):67–81, 2000.
- [2] Jinho Baik and Toufic M. Suidan. A GUE central limit theorem and universality of directed first and last passage site percolation. *Int. Math. Res. Not.*, (6):325–337, 2005.
- [3] Thierry Bodineau and James Martin. A universality property for last-passage percolation paths close to the axis. *Electron. Comm. Probab.*, 10:105–112 (electronic), 2005.
- [4] R. Durrett. *Probability: Theory and Examples*. Duxbury Press, third edition, 2005.
- [5] Nicos Georgiou. Soft edge results for longest increasing paths on the planar lattice. *Electron. Commun. Probab.*, 15:1–13, 2010.
- [6] Peter W. Glynn and Ward Whitt. Departures from many queues in series. *Ann. Appl. Probab.*, 1(4):546–572, 1991.
- [7] Janko Gravner, Craig A. Tracy, and Harold Widom. Limit theorems for height fluctuations in a class of discrete space and time growth models. *J. Statist. Phys.*, 102(5-6):1085–1132, 2001.
- [8] Janko Gravner, Craig A. Tracy, and Harold Widom. Fluctuations in the composite regime of a disordered growth model. *Comm. Math. Phys.*, 229(3):433–458, 2002.

- [9] Janko Gravner, Craig A. Tracy, and Harold Widom. A growth model in a random environment. *The Annals of Probability*, 30(3):1340–1368, 2002.
- [10] Kurt Johansson. Shape fluctuations and random matrices. *Comm. Math. Phys.*, 209(2):437–476, 2000.
- [11] Kurt Johansson. Discrete orthogonal polynomial ensembles and the Plancherel measure. *Ann. of Math. (2)*, 153(1):259–296, 2001.
- [12] Joachim Krug and Pablo Ferrari. Phase transitions in driven diffusive systems with random rates. *J. Phys. A*, 29:L465–L471, 1996.
- [13] James Martin. Limiting shape for directed percolation models. *The Annals of Probability*, 32(4):2908–2937, 2004.
- [14] Valentin V. Petrov. *Limit theorems of probability theory*, volume 4 of *Oxford Studies in Probability*. The Clarendon Press Oxford University Press, New York, 1995. Sequences of independent random variables, Oxford Science Publications.
- [15] R. Tyrrell Rockafellar. *Convex analysis*. Princeton Mathematical Series, No. 28. Princeton University Press, Princeton, N.J., 1970.
- [16] R. Tyrrell Rockafellar. *Convex analysis*. Princeton Mathematical Series, No. 28. Princeton University Press, Princeton, N.J., 1970.
- [17] H. Rost. Nonequilibrium behaviour of a many particle process: Density profile and local equilibria. *Z. Wahrsch. Verw. Gebiete*, 58(1):41–53, 1981.
- [18] Timo Seppäläinen. Increasing sequences of independent points on the planar lattice. *Ann. Appl. Probab.*, 7(4):886–898, 1997.

- [19] Timo Seppäläinen. Exact limiting shape for a simplified model of first-passage percolation on the plane. *The Annals of Probability*, 26(3):1232–1250, 1998.
- [20] Timo Seppäläinen and Joachim Krug. Hydrodynamics and platoon formation for a totally asymmetric exclusion model with particlewise disorder. *Journal of Statistical Physics*, 95(3-4):525–567, 1999.
- [21] Richard R. Weber. The interchangeability of $/m/1$ queues in series. *Journal of Applied Probability*, 16(3):690–695, 1979.